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McGRAW-HILL PUBLICATIONS IN AERONAUTICAL SCIENCE
Jerome C. Hunsaker, Consulting Editor

INSTRUMENT ENGINEERING

VOLUME I

METHODS FOR DESCRIBING THE SITUATIONS OF INSTRUMENT ENGINEERING

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Jerome C. Hunsaker, Consulting Editor

Draper, McKay, and Lees

INSTRUMENT ENGINEERING

VOL. I. METHODS FOR DESCRIBING THE SITUATIONS OF INSTRUMENT ENGINEERING.
VOL. II. METHODS FOR ASSOCIATING MATHEMATICAL SOLUTIONS WITH COMMON FORMS.
VOL. III. APPLICATIONS OF THE INSTRUMENT ENGINEERING METHOD.

Kuchemann and Weber

AERODYNAMICS OF PROPULSION

Shaw and Macks

ANALYSIS AND LUBRICATION OF BEARINGS

Streeter

FLUID DYNAMICS

Theodorsen

THEORY OF PROPELLERS

Von Mises

THEORY OF FLIGHT

INSTRUMENT ENGINEERING

VOLUME I

METHODS FOR DESCRIBING THE SITUATIONS OF INSTRUMENT ENGINEERING

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INSTRUMENT ENGINEERING

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PREFACE

Instrument Engineering is a systematic presentation of principles and methods that have been found effective during more than twenty years of teaching and practice in the field of measurement and control. These activities have been carried on in the Aircraft Instrument Section and the Instrumentation Laboratory of the Aeronautical Engineering Department at the Massachusetts Institute of Technology. The teaching has been done principally at the graduate level, for an annual enrollment during the last few years of about seventy students. The practice has included close connections with the invention, design, development, large-scale production and commercial and military applications of both simple and complex devices. Over the last decade, the personnel of the Instrumentation Laboratory has varied in size from about ten to more than four hundred members, with professional engineers accounting for approximately one-fourth of the total. Under the organizational setup used by the laboratory, a central engineering group has carried the technical responsibility for a number of projects requiring the use of knowledge from many fields. Under these circumstances, a generalized system of concepts, notation, and analysis procedures has developed. This system is not only useful to practicing engineers but also is effective in teaching engineering students the principles that have been proven useful by practical experience.

Instrument Engineering is made up of contributions by the authors, their associates and students and is based on theory from the fields of mathematics, classical physics, aeronautical engineering, mechanical engineering, electrical engineering and chemical engineering. It presents as an organized pattern the background knowledge and procedures required to solve all commonly occurring measurement and control problems. The suggested procedures are illustrated by typical examples that provide models for working out engineering problems and also supply a considerable amount of quantitative information on many situations that are important in practice.

The organization of Instrument Engineering differs from that ordinarily used in textbooks. The common textbook procedure is to start with brief elementary discussions of principles that are then developed by a series of examples. Each of the examples includes a discussion of setting up the problem involved, carrying out the mathematical reasoning necessary for the particular situation and, finally, interpretating the results. Instrument Engineering differs from this arrangement by building up in one volume a generalized pattern for describing physical situations, developing the background for associating solutions with these descriptions in a second volume, and presenting a number of systematically selected examples in a third volume.

The examples given in Volume III start off with simple measuring and indicating instruments and then progress through regulators and positional servomechanisms to analog-type simulators and elementary fire control systems. In presenting these examples, the material of Volumes I and II is used and expanded as required for the specific situation involved. This means that the discussion of any given problem may be presented concisely in Volume III, with the material of Volumes I and II supplying essential information on concepts, notation, mathematical procedures, and numerical results.

The material of Volumes I and II is selected and organized so that information is easily available to practicing engineers and to students of both graduate and undergraduate levels. A properly selected sequence of examples from Volume III, together with appropriate references to Volumes I and II, is effective in teaching instrument engineering. For this purpose, the student may be asked to apply the information given in Volume II in nondimensional form to specific cases or to work out new problems using the methods illustrated by model solutions.

Considerable detail and a limited amount of repetition in the discussion of each subject in the first two volumes is used to reduce the number of cross references required. This type of presentation results in treatments that may seem unnecessarily long at first glance, but actually are easily understood and applied in engineering and in teaching. When any part of the material in Volumes I and II has been assimilated, it may be accepted without comment in working out solutions for problems like those of Volume III. As a consequence, these solutions are given as short presentations that are complete without including more than a small amount of background material.

The background material of Volumes I and II provides a range of choices in concepts, terminology, notation, and mathematical reasoning procedures. A generalized pattern of functionally named concepts, represented by self-defining symbols and capable of immediate application to the simplest and also the most complex situations, is set up in terms of a simple scheme of conventions. Any given problem may easily be represented by means of the generalized pattern or it may be handled with the names and notation ordinarily used by workers in the engineering field in which the problem occurs. For convenience in discussing simple components or subsystems and in finding solutions for mathematical forms, working variables of minimum complexity are used. The immediate understanding of results from complicated problems is facilitated by expressing these results in self-defining symbols that may or may not be supplemented by the conventional symbols used in the engineering fields concerned. For example, when conventional notation is used in a problem involving aircraft control, the lower-case Greek letter theta (θ) may represent an angle describing one aspect of the flight path and at the same time may occur as the commonly accepted servomechanism symbol for an input or an output variable. The confusion that exists in situations of this kind, even after identifying subscripts are added to the letters involved, may be completely eliminated by the use of self-defining symbols.

Volume II draws together the common mathematical forms of instrument engineering in working variable terms that are not related to any specific physical situation. The classical, operational, and weighting-function approaches to finding solutions for these forms are drawn together in a unified picture. This method of presentation is particularly useful to engineers and students when they deal with groups of colleagues who are trained to use only one approach to the mathematical treatment of physical problems. Familiarity with several mathematical viewpoints makes it possible to select the most effective method for dealing with any given situation. For a reason of the same kind, Volume II describes not only conventional differential equation methods but also several graphical techniques such as the frequency-response, roots-locus, and numerical procedures for associating solutions with mathematical forms. Uses for all of these procedures are illustrated in the examples of Volume III.

The authors are indebted to many sources for the background material of Instrument Engineering. The basic mathematical references have been the texts on advanced calculus by Philip Franklin and by F. B. Hildebrand, and the works on ordinary differential equations by E. L. Ince, by H. B. Phillips, and by Abraham Cohen. The techniques of the operational calculus have been drawn from the books by M. F. Gardner and J. L. Barnes, by H. S. Carslaw and J. C. Jaeger, and by R. V. Churchill. The recently published book, The Mathematics of Circuit Analysis, by E. A. Guillemin has served as the basis for the discussion of stability of mathematical forms. The notions of differential geometry and tensor analysis taught by Professor D. J. Struik of the Mathematics Department, Massachusetts Institute of Technology, have suggested the conventions for self-defining notation. Publications on the technique of the roots-locus method by W. R. Evans, and unpublished work by R. T. Donohue of the Instrumentation Laboratory staff, have supplied the background for this approach to the study of the dynamic performance of systems. Network Analysis and Feedback Amplifier Design by H. W. Bode; articles by A. C. Hall and by H. T. Harris on servomechanism theory; Theory of Servomechanisms by H. M. James, N. B. Nichols and R. S. Phillips; Principles of Servomechanisms by G. S. Brown and D. P. Campbell; Servomechanism

Fundamentals by Henri Lauer, Robert Lesnick and L. E. Matson; and various other treatises in this field have been used as the basic references for the application of frequency analysis methods to feedback systems.

In order to include essential information developed not by the authors but by other members of the Instrument Section, several chapters in this book have been specially prepared by engineers who receive direct credit at the beginning of their contributions.

Instrument Engineering will serve as a textbook for students and at the same time be a basic reference for engineers who have either continuous or infrequent contacts with the subject. In its present form, it is fundamental and complete enough to serve as the basis for both expanded and abridged treatments of an area that is taking its place among the established engineering fields. The authors commend Instrument Engineering to its readers with the hope that it will not only be helpful in solving current problems but also will provide inspiration for developments that will appear in the future.

C. S. Draper
W. McKay
S. Lees

Cambridge, Massachusetts
April, 1952

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INTRODUCTION

GENERAL NATURE OF INSTRUMENT ENGINEERING

1. Instrument engineering is the art and science by which the properties of matter and the laws of nature are made useful to man in the observation and control of physical processes. This definition for the subject matter of this book implies some modifications of the meanings given for the terms instrument* and engineering** in WEBSTER'S COLLEGIATE DICTIONARY. In this reference, the word instrument is applied to "that by means of which something is performed or effected . . .," implying ". . . a more delicate operation . . . as surgical instrument, astronomical instrument, a draftsman's instruments" . . . and ". . . personal manipulation. . . ." Engineering as defined by Webster is ". . . the art and science by which the properties of matter and the sources of power in nature are made useful to man in structures, machines, and manufactured products."

2. Instruments that are the basic subject matter for this book are not intended for "personal manipulation" in the sense that the violin or the surgical scalpel are used by human beings. Rather, they are devices for the observation and control of physical quantities*** without any primary dependence upon personal manipulation. Observation implies that indications, which may be either index positions or some equivalent phenomenon, or measurements in numerical terms, are associated with physical situations. Control implies that physical

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* Webster's Collegiate Dictionary gives for instrument: 1. That by means of which something is performed or effected; a medium, means. 2. A tool; utensil; implement. 3. A contrivance by which musical sounds are produced. 4. Law. A writing as a deed, writ, etc. Syn. See Implement.

Implement. . . An article of equipment; esp. a tool, utensil, instrument, etc., essential to the performance or execution of something. Syn. Implement, tool, utensil, instrument suggest relatively simple construction and personal manipulation. Implement and tool are often interchangeable. But implement is the broader term, often implying that by which any operation is carried on; tool commonly suggests the implement of a craftsman or laborer. Utensil applies esp. to the implements of the kitchen; as cooking utensils. Instrument implies more delicate operations than tool; as surgical instruments, astronomical instruments, a draftsman's instruments. Instrument and tool (the latter in this sense always a term of contempt) are applied figuratively to one made use of for another's purpose.

** Engineer. 1. A designer or constructor of engines. 2. Mil. and Naval. One of a corps of men who perform engineering work; as in building forts, bridges, etc. 3. One versed in, or who follows as a calling, any branch of engineering. 4. One who operates an engine.

** Engineering. 1. Originally, the art of managing engines; now, the art and science by which the properties of matter and sources of power in nature are made useful to man in structures, machines and manufactured products.

*** By definition, a physical quantity is any thing or state of things that may be described in terms of numbers.

quantities are caused to change in accordance with commands that may be either generated automatically or imposed by independently-made manual settings.

3. Gages that receive pressure and produce indications by means of pointers moving over scales are examples of measuring instruments. Regulators that receive settings from external sources and automatically adjust the temperatures of controlled bodies to these settings are control instruments. Automatic pilots that receive settings of altitude, course, speed, etc., and make the required adjustments for keeping aircraft flying in accordance with these commands are control systems that include many instruments in their subsystems.

4. The examples of the preceding paragraph are typical of the devices that fall into the class of instruments or instrument systems. The common feature of all indicating and measuring instruments is that they generate information on physical quantities. The corresponding common function of control instrument systems is that they receive, transmit, process, and use information on physical quantities for the control of these quantities.

5. Because instrument systems process information rather than process or produce materials or energy, they are generally designed to operate at the lowest power levels consistent with satisfactory performance. This means that effectiveness, rather than efficiency, is the primary consideration for instrument system design.

6. In general, the effectiveness required of instrument systems must be realized by combining techniques from many fields of science and engineering. It is the considerable background of knowledge and skill that must be drawn from a number of usually unrelated regions that has resulted in the development of instrument engineering as an entity* that cuts across the boundaries of traditional fields. A large amount of technical knowledge is readily available to the instrument engineer but, because this information is drawn from many independently developed regions, he is confronted with a great confusion in concepts, terminology, and symbols.** The confusion is increased because various branches of engineering use apparently different treatments of identical problems.

7. The instrument engineer must be able to coordinate all the viewpoints involved in any given measurement or control problem so that each is given sympathetic and intelligent handling. He must understand and respect the achievements in the areas with which he has contact and be able to effectively discuss ideas in the special languages of these fields. At the same time, he must

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* Entity . . . a thing which has reality and distinctness of being either in fact or for thought . . .

** This is a common problem. For example, Prof. Norbert Wiener, in his book Cybernetics, John Wiley & Sons, Inc., New York, 1948, states, ". . . we were seriously hampered by the lack of unity of the literature concerning these problems, and by the absence of any common terminology, or even of a single name for the field."

have a system of concepts and techniques expressed in his own generalized language and designed to serve as a common denominator for the correlation of information from many sources. In essence, this does not mean that the instrument engineer is a parasite who depends entirely on material from other fields. Rather, it means that he is a cosmopolitan who is a citizen in the world of science and engineering and has a language of his own, but who is an accomplished linguist with genuine understanding and sympathy for the ideas of his colleagues. In many situations, the instrument engineer must also be a diplomat able to achieve his goal of a satisfactory working system by persuasion rather than by the exercise of actual authority over all phases of an installation for which he must provide measurement and control.

SCOPE OF INSTRUMENT ENGINEERING

8. Instrument engineering includes the actual carrying out or supervision of the design, installation, maintenance, and use of instrument systems, including all types of measurement and control equipment. Two important phases in this work are the theoretical analysis of performance and the evaluation and interpretation of results in terms of performance requirements.

9. In addition to system applications which may involve only the combination of existing components, instrument engineering includes the development of new measurement and control systems, beginning with performance requirements and ending in the manufacture of commercially satisfactory products. The engineer must clearly understand that projects of this type separate naturally into the three phases* shown in the information flow diagram of Fig. 1. These phases are:

1) Research and development phase
2) Production design development phase
3) Production phase

The research and development phase is initiated by the receipt of performance requirements, and generates as its primary output the information necessary to carry out the production design. The secondary output is design information for test and inspection equipment. This information is received by the production design development phase, which produces manufacturing information for the product and for the test and inspection equipment. The output from the production phase is the desired product. Ordinarily, test and inspection equipment is used in the manufacturing procedure and appears only incidentally as an output of the production phase.

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* For the purposes of the present discussion, only technical phases are considered. In practice, contractual relationships, accounting organization, personnel organization, and policy supervision must all be properly coordinated with the engineering work if a satisfactory result is to be achieved.

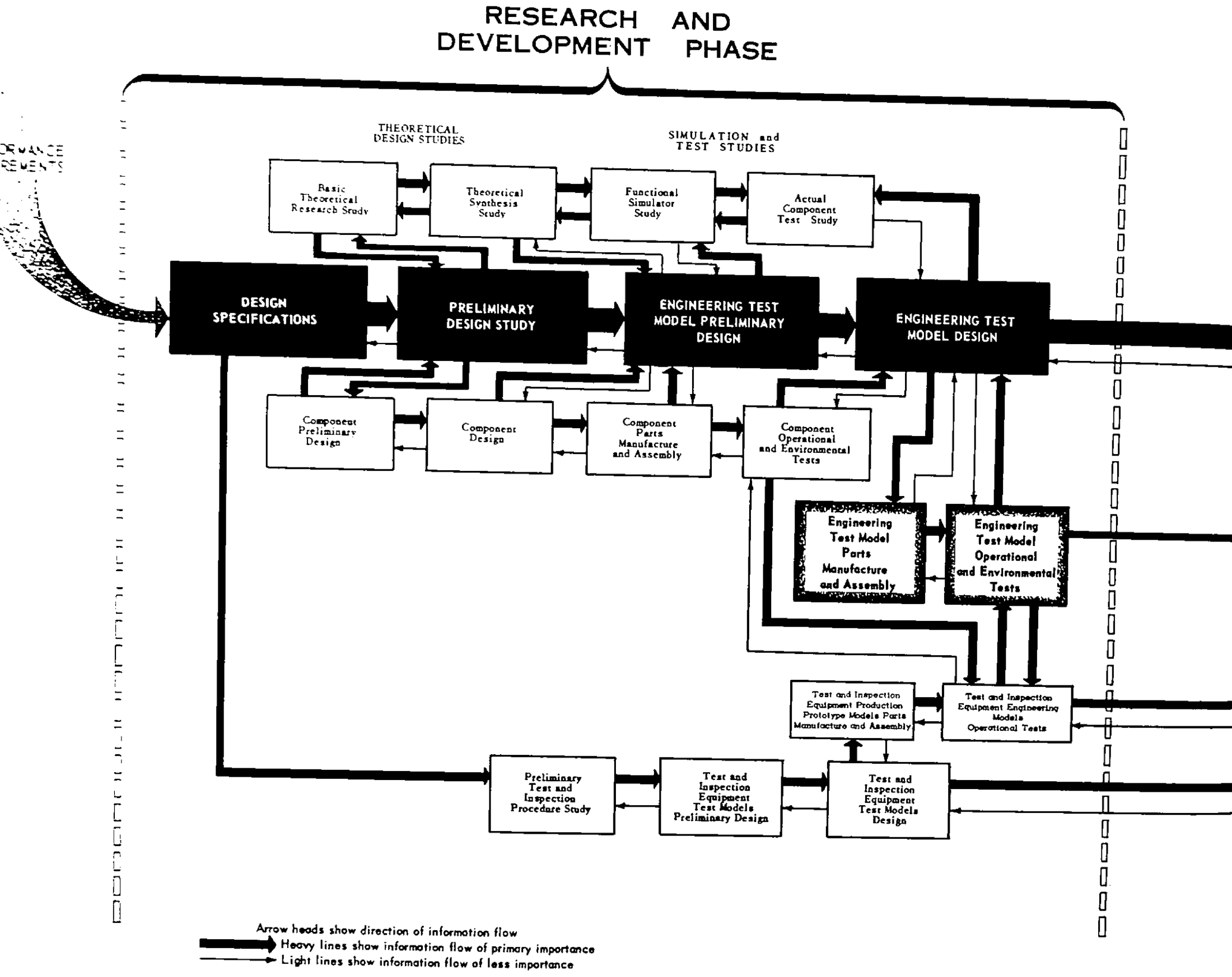
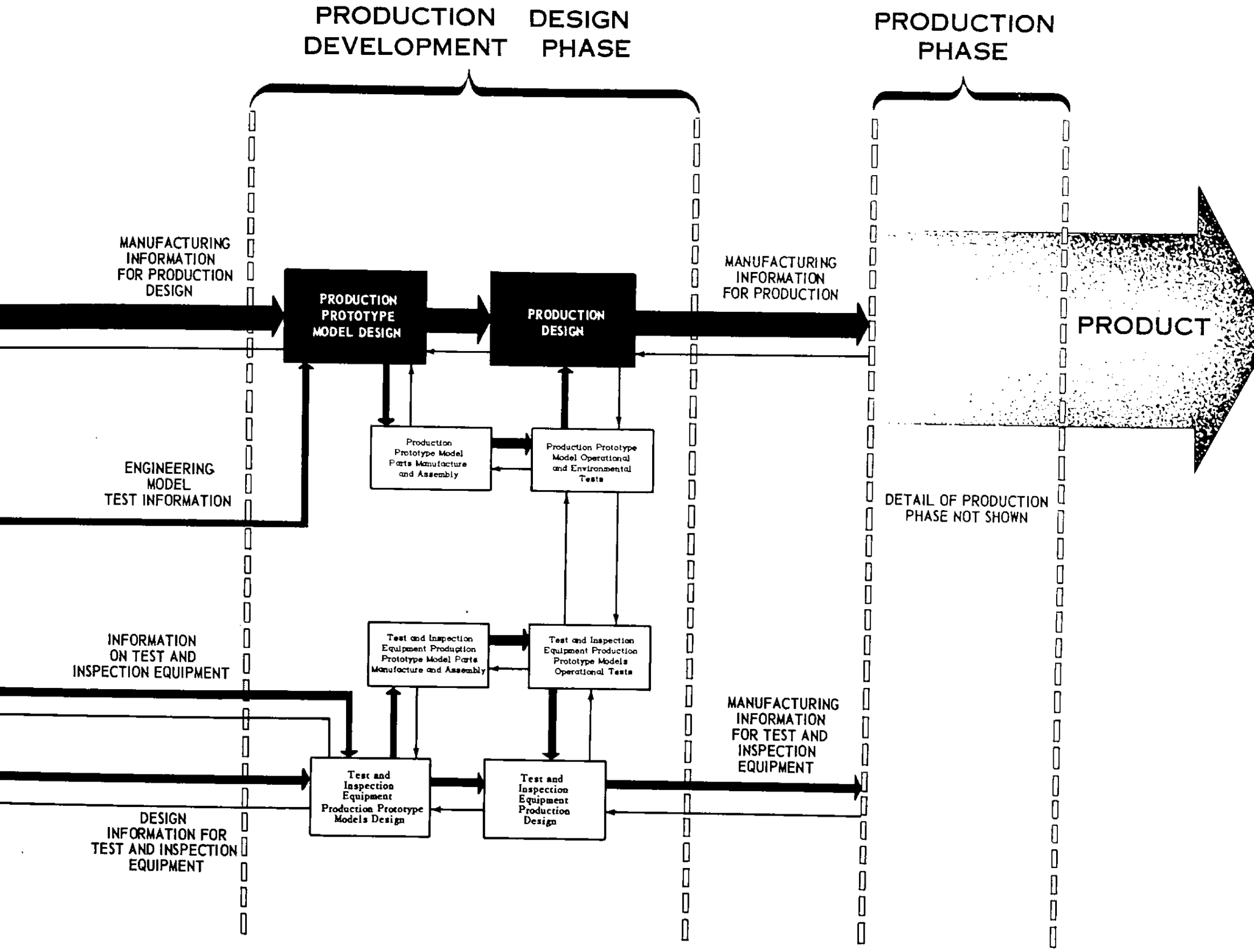

RESEARCH AND DEVELOPMENT PHASE
THEORETICAL DESIGN STUDIES
SIMULATION and TEST STUDIES
Basic Theoretical Research Study
Theoretical Synthesis Study
Functional Simulator Study
Actual Component Test Study
DESIGN SPECIFICATIONS
PRELIMINARY DESIGN STUDY
ENGINEERING TEST MODEL PRELIMINARY DESIGN
ENGINEERING TEST MODEL DESIGN
Component Preliminary Design
Component Design
Component Parts Manufacture and Assembly
Component Operational and Environmental Tests
Engineering Test Model Parts Manufacture and Assembly
Engineering Test Model Operational and Environmental Tests
Test and Inspection Equipment Production Prototype Models Parts Manufacture and Assembly
Test and Inspection Equipment Engineering Models Operational Tests
Preliminary Test and Inspection Procedure Study
Test and Inspection Equipment Test Models Preliminary Design
Test and Inspection Equipment Test Models Design
Arrow heads show direction of information flow
Heavy lines show information flow of primary importance
Light lines show information flow of less importance

FIG. 1. INFORMATION FLOW DIAGRAM FOR


PRODUCTION DEVELOPMENT
DESIGN PHASE
PRODUCTION PHASE
MANUFACTURING INFORMATION FOR PRODUCTION DESIGN
PRODUCTION PROTOTYPE MODEL DESIGN
PRODUCTION DESIGN
MANUFACTURING INFORMATION FOR PRODUCTION
PRODUCT
Production Prototype Model Parts Manufacture and Assembly
Production Prototype Model Operational and Environmental Tests
ENGINEERING MODEL TEST INFORMATION
DETAIL OF PRODUCTION PHASE NOT SHOWN
INFORMATION ON TEST AND INSPECTION EQUIPMENT
Test and Inspection Equipment Production Prototype Model Parts Manufacture and Assembly
Test and Inspection Equipment Production Prototype Models Operational Tests
MANUFACTURING INFORMATION FOR TEST AND INSPECTION EQUIPMENT
Test and Inspection Equipment Production Prototype Models Design
Test and Inspection Equipment Production Design
DESIGN INFORMATION FOR TEST AND INSPECTION EQUIPMENT

THE DEVELOPMENT OF AN INSTRUMENT SYSTEM

10. The information flow diagram of Fig. 1 shows a primary chain of boxes which represent the phases through which a design must pass from the stage of specifications until it is satisfactory for use in production. This primary chain is supported by three parallel chains. In the flow diagram, an upper chain of theoretical design, simulation, and test studies receives data from the preliminary design study and supplies information for the engineering test model preliminary design. Arrows in the general direction from the performance requirements as the input to the product as the output show the transfer of information along the "direct" path. "Feedback" information is represented by arrows drawn in the opposite direction. Heavy arrows show information transfer of primary importance. Lighter arrows indicate information transfer of less importance. The component and system test chain appears in the flow diagram just below the design chain. This parallel chain represents the generation of information to support design work by actual tests of experimental models. The requirement for free exchange of data among all the boxes of the test chain and the design chain is indicated by vertical arrows in the flow diagram. The test and inspection equipment development chain appears as the bottom line in the flow diagram. This chain has an auxiliary parallel chain which represents test work on actual equipment. The diagram shows the strong interchange of information between the test equipment chain and the test chain.

11. In practice, the only object of the research and development phase of a project is to generate information from which a production design may be developed. This means that the work of a limited number of able engineers should be assisted in every possible way by allowing them to operate with proper facilities under rules designed for flexibility and the rapid translation of ideas into theoretical results and working test models. It is of primary importance that all information supplied for production design purposes be checked by experimental results from a working test model. This test model should have substantially all the functional features of the proposed product, but, in general, will be hand-made in the shop rather than fabricated with production techniques.

12. In the production design development phase of the information flow diagram, all the rules, procedures, and restrictions required in successful production should be applied to the design work. All designs should be checked by experimental tests on a small number of prototype models to be sure that the essential functions are preserved when all the production design features are introduced. It is important that trials of the prototype model be carried out with the test equipment proposed for use in actual production.

13. The engineering procedure shown in the information flow diagram for the development of instrument systems represents the ideal situation. Any actual development will follow a pattern more or less different from that of the flow

diagram. In some cases of small production, the research and development phase may be stressed to a point where each production item is treated as an engineering model. On the other hand, stress may be removed from the research and development phase to such an extent that it becomes necessary to carry out the research function by making changes in production lines. This procedure is likely to be slow and expensive, but it must be recognized as one way of generating the manufacturing information required for a satisfactory output item.

14. It is obviously impossible for the instrument engineer to qualify as a professional expert in all details of the development procedure suggested by the information flow diagram. However, it is important for him to know the general character of the interactions between the various steps in the required processes and, in particular, to realize the necessity for making sure that all informational loops to the main design chain are closed by proper feedback channels from activities that check and extend design information. The instrument engineer should be able to recognize deficiencies in the information itself or in the channels for its transfer and to remove these deficiencies either by direct analysis carried out by himself and his own group or by coordination of the efforts of a cooperating organization. To be effective in this area, the instrument engineer not only must have a broad viewpoint and training in a number of fields but also should develop a sufficient feeling for the opinions of others to know when he is an effective coordinator and when he has stepped across the boundary into the region of becoming obnoxious by "minding other people's business." The formal training may be acquired by any individual with an engineering or science background who is willing to make the required effort over and beyond that necessary to take an engineering degree in any field. On the other hand, the ability to effectively assist in the coordination of system work without holding firm over-all authority is something that will be acquired by only a limited number of individuals. However, in any case, the instrument engineer should have enough of the over-all viewpoint for him to realize that satisfactory results depend upon the successful operation of all components of a system rather than the superlative performance of the few parts that may be under his direct control.

THE METHOD OF INSTRUMENT ENGINEERING

15. The basic problem of instrument engineering is to determine the effects of instruments and controls, which in general belong to the class of operating systems, in establishing functional relationships among physical quantities. As a general rule certain quantities will be independent of any actions of some operating systems, so that they belong in the class of inputs for these systems. Other quantities are outputs in the sense that they depend not only on the inputs but also on the operating systems involved. Formally, one part of the required result in

any given situation is a solution giving the response in terms of output variations that depend upon the input variations to the operating system. In order for a solution to be of practical use for engineering purposes, the essential physical facts associated with the operating system considered must be described in terms of diagrams, figures, uniquely defined symbols, and mathematical forms. The required solutions may be found by the use of mathematical processes, computers, simulators, or any other available means. With solutions available, the remaining step in an engineering problem is to interpret the results in terms of the existing physical situation. In practice, many variations of the basic problem will appear, but it is always of primary importance to express the performance characteristics of operating components* and systems in terms of performance-describing concepts especially adapted for use in measurement and control problems. Performance characteristics for a number of commonly occurring operating component types are given in later chapters.

16. The plan followed in this treatment is to describe the material and develop the methods of instrument engineering in three major parts, each of which is the primary subject matter of a volume:

I. METHODS FOR DESCRIBING THE SITUATIONS OF INSTRUMENT ENGINEERING.

II. METHODS FOR ASSOCIATING MATHEMATICAL SOLUTIONS WITH COMMON FORMS.

III. APPLICATIONS OF THE INSTRUMENT ENGINEERING METHOD.

17. The first volume deals with the essential physical facts associated with problems of measurement and control and the techniques by which these facts may be represented for purposes of description and analysis. Description implies first the use of proper concepts to define the physical quantities involved and symbols to represent the quantities and the concepts. Similarly, concepts and symbols must be defined to describe the performance of all operating systems and their components.

18. With concepts and symbols available, it is necessary to show interrelations among physical quantities and operating components by diagrams and other means. Diagrams of many types are required, ranging from cutaway drawings, through pictorial drawings and pictorial schematics, to the functional and mathematical diagrams used for analysis and design purposes.

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* Operating components are the "building blocks" of which operating systems are formed. By definition, operating components establish functional relationships between their input quantities and their corresponding output quantities.

19. Mathematical expressions to describe the performance of operating systems and their components may be developed from the information summarized in mathematical diagrams. These expressions are derived by applying natural laws to the components of operating systems. The equations formed by this procedure are the basic material for defining performance-describing concepts that may be reduced to standard forms specially adapted to the purposes of instrument engineering. The fundamental performance-describing concept is the performance operator, which expresses in quantitative terms the action of a given operating component or operating system in establishing functional relationships among the essential inputs and outputs. Various modifications of the performance operator concept will appear in the discussions of Volume I.

20. A survey of the equations required to describe the performance of operating systems for the purposes of instrument engineering discloses certain general patterns. These similarities make it possible to set up generalized forms that effectively express the performance of substantially all operating systems that are of engineering interest. Solutions of these forms for several simple but highly important cases and also for situations of indefinite complexity are the subject matter for Volume II.

21. Volume II takes as its basic problem the finding of solutions for generalized performance-describing equations that may be developed by the methods of Volume I. A widely useful performance-describing form is the linear differential equation with constant coefficients. Complete solutions of this form are possible when the equation is of first or second order. Solutions for higher order forms are also possible, but the methods required are so cumbersome that special graphical and computational procedures are required. In these procedures, it is convenient to analyze equations of any order into combinations of first- and second-order forms. For this reason, it is important to have immediately available generalized data on first- and second-order solutions. These data are developed in Volume II and are summarized both graphically and in numerical tabulations. Volume II also includes a discussion of stability in solutions of equations of arbitrary order with special attention to equations of seventh order and less. The stability problem is treated in terms of various analytical and graphical methods and is coordinated with other aspects of finding solutions for differential equations. The results developed in Volume II are presented in nondimensional terms which make them immediately applicable to any problem that may be described by certain easily identified equation forms. This immediate availability of information is especially valuable for the synthesis phases of instrument system design and is helpful for the purposes of analysis and data evaluation.

22. Volume III is made up of examples in which the methods and information developed in Volumes I and II are used to solve various problems of instrument

engineering. The examples of Volume III are chosen because they represent common situations for which it is desirable to have solutions immediately available and because they illustrate typical applications of engineering principles.

23. The situations considered in Volume III progress in complexity from measuring instruments that may be described by first-order differential equations to control systems using feedback chains with variable characteristics. Solutions for this range of examples require the use of all the analytical and graphical methods developed in Volume II and suggest procedures to be used in treating any problems that may occur in practice. Open-chain systems and closed-chain systems, including typical servomechanisms, are intermixed without formal distinctions between various types. Classifications are made on the basis of complexity of performance equations, not by the organization of the system which is described. Volume III is intended to be a ready source of quantitative performance information on typical instrument systems and to concisely illustrate the instrument engineering approach to problems of all types.

CHAPTER 1

THE OPERATING SYSTEM

A. THE OPERATING COMPONENT

1.1. Instrument engineering is based on the concept of the operating component, which, by definition, establishes functional relationships among the physical quantities that act as its inputs and its outputs.* A typical example of an operating component is pictured in Fig. 1-1, which shows a simple spring-and-index combination. The input to the operating component is a force — the force that stretches the spring. The output is the index displacement, which results from the stretching of the spring.

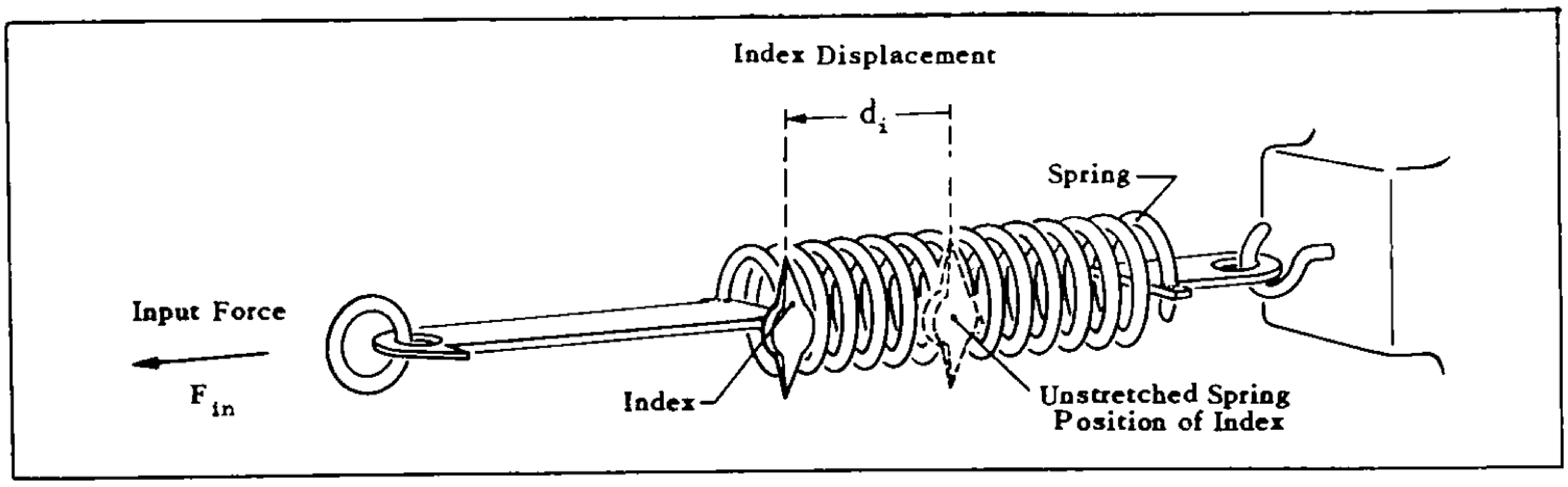

Fig. 1-1. Spring-and-index combination as an operating system with a single input and a single output.

1.2. While Fig. 1-1 shows the physical features of the operating component, its functional nature is more readily described graphically in the way shown in Fig. 1-2a, which is a functional diagram for the spring-and-index combination. The form of the functional diagram is applicable to any operating component, as shown in Fig. 1-2b. Typically, a rectangular box represents the operating component. For convenience in making written reference to the component, some form of abbreviated coding is frequently desirable. The letters (oc) suggest themselves for operating component, and similarly the letters (s + i) serve to identify spring-plus-index, the enclosing parentheses designating the letters as a single expression.

1.3. The input is shown entering the operating component from the left, the arrowhead fixing the direction of flow of the input quantity. Again, a symbol is generally useful in referring to this quantity. The letter q is used to represent a physical quantity. To show that q is an input quantity, a subscript in is used to form the complete symbol. Similarly, in the functional representation of the spring-and-index combination, the input force is represented by the more quickly written F_{in}. The same treatment is extended to the output of the component, which becomes q_{out} for the generalized form, and d_i for the displacement of the index.

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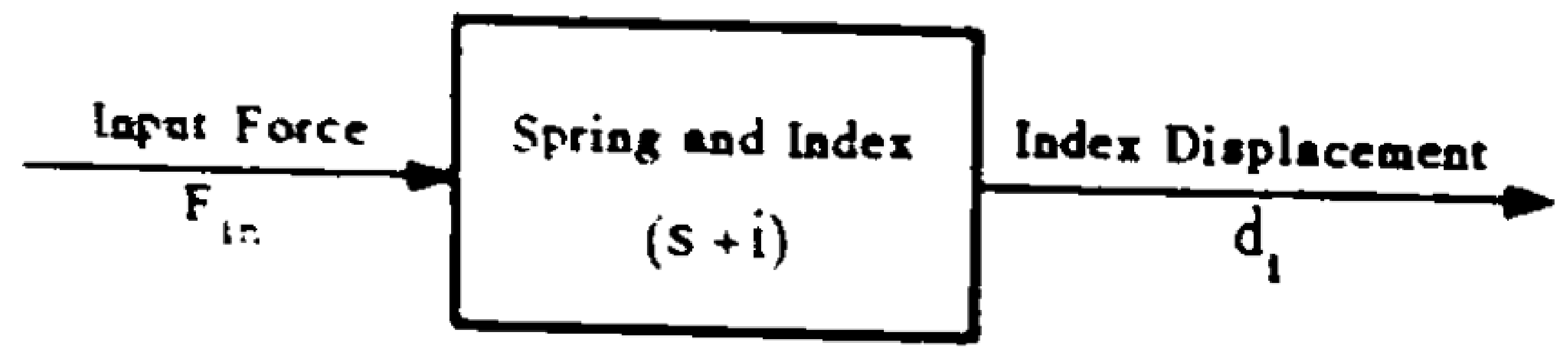
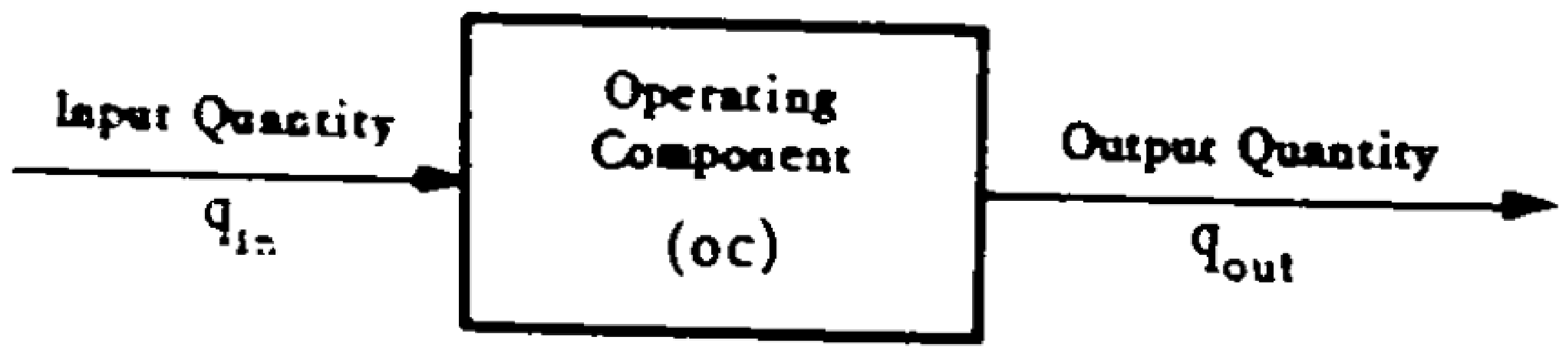
* In general, inputs and outputs are quantities that vary with time. This dependence on time is not explicitly noted except where it is necessary for the treatment of particular problems.



a) Functional diagram for a spring-and-index combination



b) Generalized functional diagram for an operating component

Fig. 1-2. Functional diagrams for an operating component with a single input and a single output.

1.4. The introduction to symbolic notation given here for an operating component is extended in later chapters, as discussions require, to include all phases of instrument engineering.

B. THE OPEN-CHAIN OPERATING SYSTEM

1.5. An operating system is formed by the combination of two or more operating components. For example, if a scale is added as a second operating component for the spring-and-index combination of Fig. 1-1, the result is the spring balance shown in Fig. 1-3. In this figure, the scale acts as an operating component in the sense that it receives the index displacement as its input and produces the force scale reading (F for force, with subscript s to show that it comes from the scale) as its output. Figure 1-4 is a functional diagram showing the two

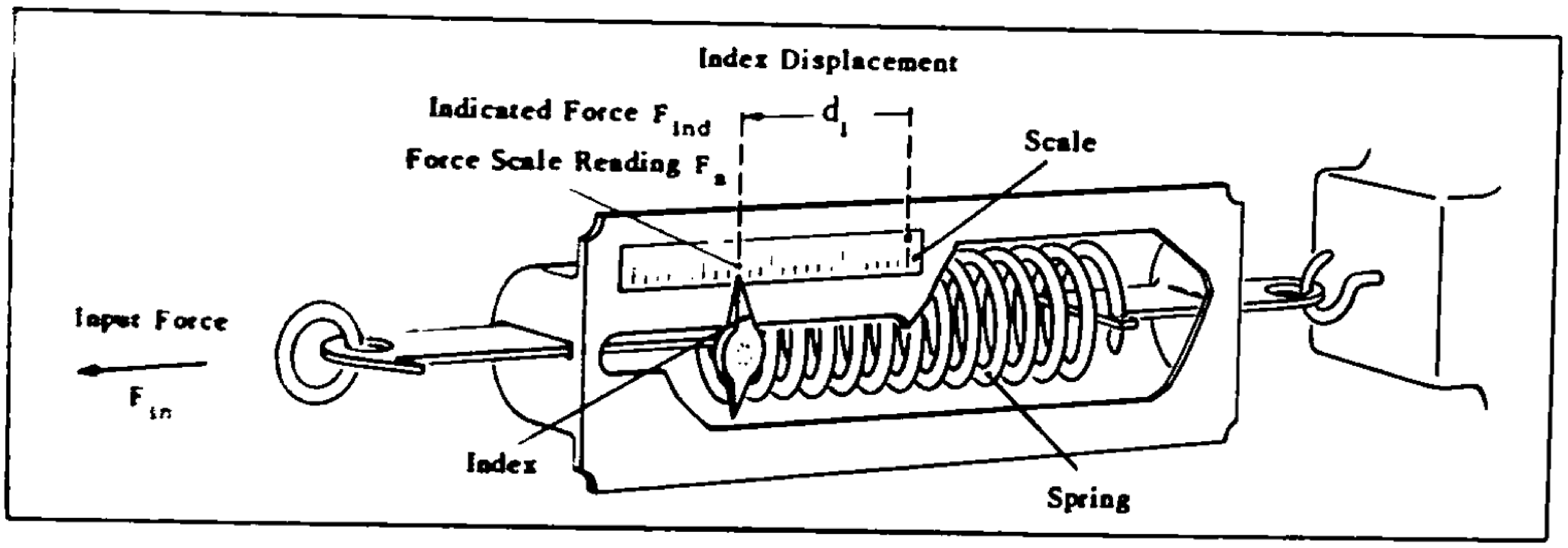


Fig. 1-3. Spring balance as an operating system.

operating components of Fig. 1-3 represented as an *open-chain system*. The term *chain* is used here because the operating components are interconnected as the links of a chain. The chain is an *open chain* because each link is connected only to the links which precede and follow it, and does not "jump across" any links either forward or backward along the chain. Figure 1-5 is a representation of an open-chain system with an indefinite number of links. It is conventional to show the input quantity entering the diagram at the left and the output quantity leaving the diagram at the right, with the same sense preserved for all intermediate links. The terminology and symbols follow the pattern described for Fig. 1-2, with the exception that subscript numbers are used to identify operating components and their associated physical quantities in order along the chain. In discussions of open-chain systems, the direction from the system input quantity to the system output quantity is called the *direct sense* along the chain. The opposite direction, from system output to system input, is called the *inverse sense* along the chain.

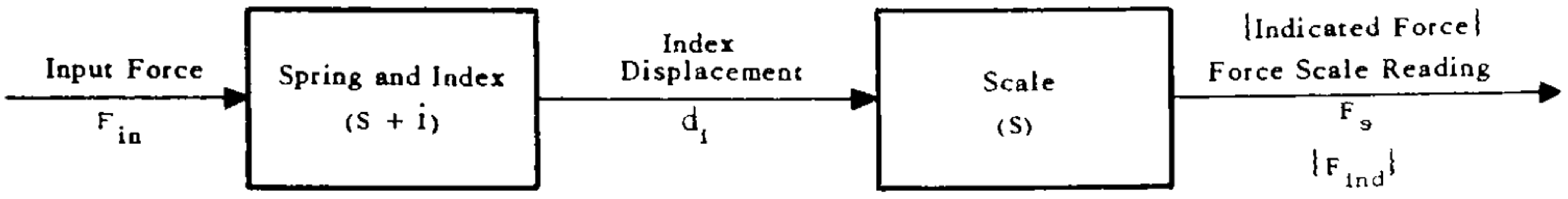


Fig. 1-4. Functional diagram for a spring balance as an open-chain operating system.

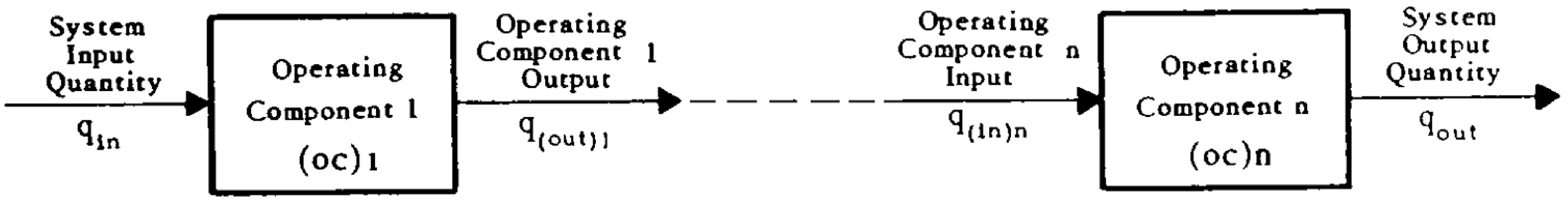


Fig. 1-5. Generalized functional diagram for an open-chain operating system.

C. THE CLOSED-CHAIN OPERATING SYSTEM

DEFINITIONS

1.6. An operating system becomes a *closed-chain system* when the output from an operating component is supplied as an input to one or more components forward or backward along the chain. The *feedback system* is a special case of the closed-chain system, in which the output from one or more operating components is supplied as an input to one or more of the preceding components in the direct sense of the main operating chain. Any parallel chain branch that transmits action in the inverse direction with respect to the main chain branch is a *feedback branch*, and the quantities it supplies as inputs to chain components are called *feedback quantities*.

FEEDBACK

1.7. Feedback branches are the distinguishing characteristic of all servomechanisms* and many control systems. In effect, feedback makes it possible for an operating system to check directly on its own operation by comparing the actual output with the correct output for a given input, and to make required corrections on the basis of differences between some function of the actual output and the input. Feedback is also useful in systems designed for automatic operation, in which it merely serves to eliminate a human operator. An application of this type is illustrated in Figs. 1-6, 1-7, 1-8, and 1-9, which describe a force measuring scale, first with manual adjustment of the balance mass and then with a feedback branch arrangement for automatically maintaining the required balance.

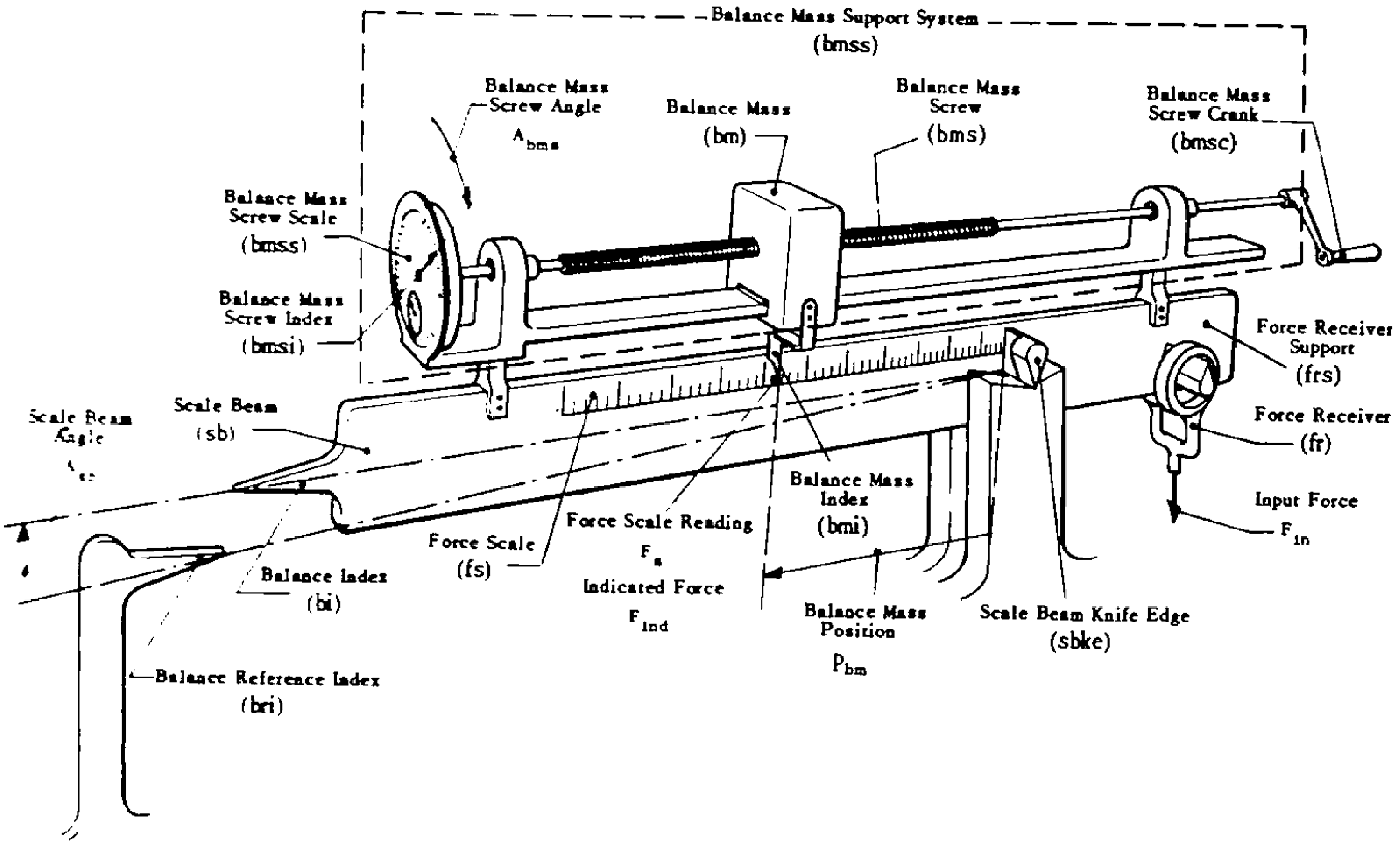


Fig. 1-6. Pictorial diagram of a force measuring scale with manual balancing adjustment.

NAMES FOR OPERATING COMPONENTS

1.8. The illustrative example of Fig. 1-6 will be used to demonstrate a generally useful scheme for naming operating components and forming associated symbols. In the procedure to be described, components are given both functional names and object names; that is, they may be designated by words that describe what they do and also by names that tell what they are. Symbols for operating components are formed** by using the initials of the assigned names as a group of lower-case letters.

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* Servomechanisms are discussed by many writers as the basis for a distinct field of engineering. See, for example, bibliographical references (1), (2), (5), (9), (15), (20), (23), (24), (27), (28), (32).

** Exceptions to this rule are made as required by special situations.

MANUAL SCALE BALANCING

1.9. The force measuring scale illustrated in the diagram of Fig. 1-6 is used to associate numbers with the input by causing the force to apply a torque component to a rigid scale beam. This scale beam also receives a torque in the opposite direction from a balance mass which is moved along the scale beam by means of a balance mass screw. The equality of the balance mass torque and the force input torque is shown by coincidence of the balance index with the balance reference index. When these two indices are not in perfect alignment, the scale beam angle (main symbol A for angle with subscript sb for scale beam) indicates the direction and amount by which the balance mass is out of adjustment. With the scale in balance so that the scale beam angle is zero, the balance mass position is determined by the force input. This position is indicated by the balance mass screw index. The indicated force may be read directly from this index or from the force scale on the scale beam. Manual adjustment of the balance mass is required when force measurements are made with the arrangement of Fig. 1-6; that is, a human operator observes the scale beam angle and adjusts the balance mass position by changing the balance mass screw angle with the balance mass screw crank. This adjustment is continued by a trial-and-error process until the scale beam angle is substantially zero. When this condition exists, the force indication may be read directly from the scale on the beam. It is important to note that although the manually-adjusted force measuring scale of Fig. 1-6 does not include any feedback arrangement within itself, no force reading is possible unless a human operator provides the feedback function by observing the scale beam angle and making the necessary corrective changes to the balance mass screw angle setting. This means that a human operator is an essential part of the measuring system, although he does not appear in the diagram.

FUNCTIONAL DIAGRAM FOR MANUALLY-ADJUSTED SCALE

1.10. Figure 1-7 is a functional diagram for the manually-adjusted force measuring scale of Fig. 1-6. With the pictorial diagram of the arrangement available, the functional diagram of Fig. 1-7 needs little explanation. In this example, the scale is an instrument of the quantity balance type in which torque is the balancing variable. The scale beam is a rigid body which receives and equates the force input torque to the balance mass torque. Any deviation from equality of these two torque components is indicated by a scale beam angle different from zero. In Fig. 1-7, two parallel operating chains generate the torque components that are equated by the torque summing member. Both of these chains are direct chains because they receive inputs and produce outputs without feedback action of any kind. One chain consists of the force receiver, which receives the force input and transmits it unchanged to the force receiver support, and then applies the input force torque to the torque summing member. The force receiver support acts as a rigid arm that multiplies the input force to produce its output torque component. The second direct chain consists of the balance mass screw, which receives the balance mass screw angle setting as an independent input and produces the balance mass position as its output. This position is one of the inputs for the balance mass support system, which receives the weight of the balance mass as its second input and effectively

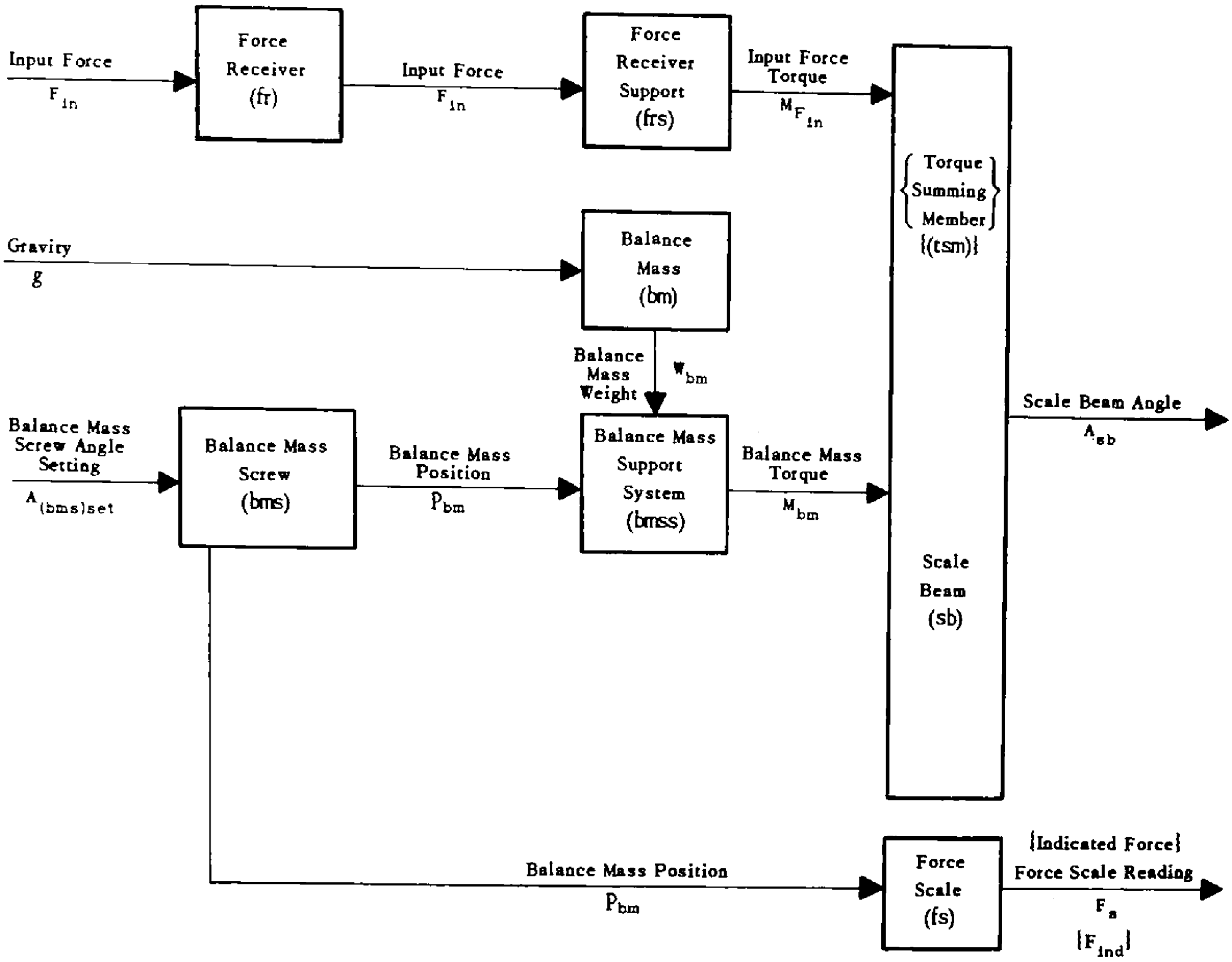

Fig. 1-7. Functional diagram of a force measuring scale with manual balancing adjustment.

multiplies the two inputs together to produce the balance mass torque as its output. The primary output of the force measuring scale is the balance mass position, which is taken from the balance mass screw and supplied as the input for the force scale. The output from the force scale is the force scale reading. In practice this reading is not a true measure of the force input unless the balance mass screw angle setting is manipulated to maintain the scale beam angle at zero. For this reason, the force measuring scale of Figs. 1-6 and 1-7 is not a self-contained instrument.

FUNCTIONAL DIAGRAM FOR AUTOMATIC BALANCING SCALE

1.11. The arrangement of Figs. 1-6 and 1-7 may be made to measure force automatically by closing the loop between the scale beam angle and the balance mass screw setting with a feedback chain of operating components. Figure 1-8 is a pictorial diagram for the force measuring scale with automatic balance adjustment. A potentiometer winding and slider arrangement is used to replace the balance index and the reference index of Fig. 1-6. The potentiometer acts as a signal generator, which receives the scale beam angle and produces an electrical signal suitable for controlling the balance mass screw drive system. This drive system consists of the balance mass screw drive motor and the balance mass screw drive motor power control system, which receives the output of the signal generator and supplies the proper power input for the drive motor. Figure 1-9 shows that the signal generator and the balance mass screw drive system form a feedback branch acting inversely in parallel with the balance mass chain shown in Fig. 1-7. Aside from the addition of this feedback branch, the essential difference between the diagram of Fig. 1-9 and the diagram of Fig. 1-7 is that in the former

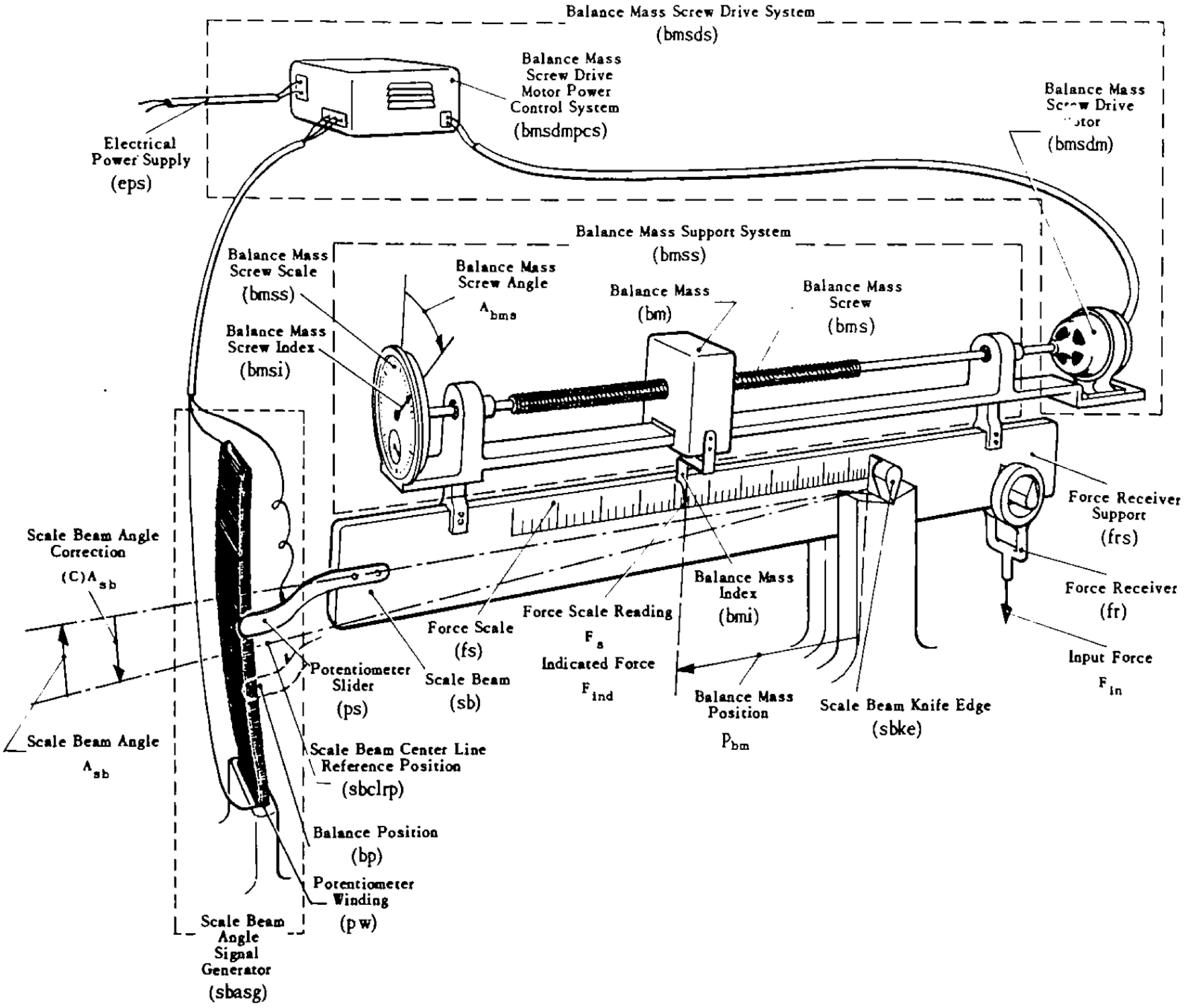
Balance Mass Screw Drive System (bmsds)
Balance Mass Screw Drive Motor Power Control System (bmsdmpcs)
Balance Mass Screw Drive Motor (bmsdm)
Electrical Power Supply (eps)
Balance Mass Support System (bmss)
Balance Mass Screw Scale (bmss)
Balance Mass Screw Angle A_{bms}
Balance Mass (bm)
Balance Mass Screw (bms)
Balance Mass Screw Index (bmsi)
Scale Beam Angle Correction $(C)A_{sb}$
Scale Beam Angle A_{sb}
Force Receiver Support (frs)
Force Receiver (fr)
Input Force F_{in}
Balance Mass Index (bmi)
Force Scale Reading F_s
Indicated Force F_{ind}
Force Scale (fs)
Scale Beam (sb)
Potentiometer Slider (ps)
Balance Mass Position P_{bm}
Scale Beam Knife Edge (sbke)
Scale Beam Center Line Reference Position (sbclrp)
Balance Position (bp)
Potentiometer Winding (pw)
Scale Beam Angle Signal Generator (sbasg)

Fig. 1-8. Pictorial diagram of a force measuring scale with automatic balancing adjustment.

figure the balance mass screw angle setting as an input and the scale beam angle as an output are both absent. This means that the adjustment function required of a human operator in Fig. 1-7 is carried out automatically in the arrangement of Fig. 1-9.

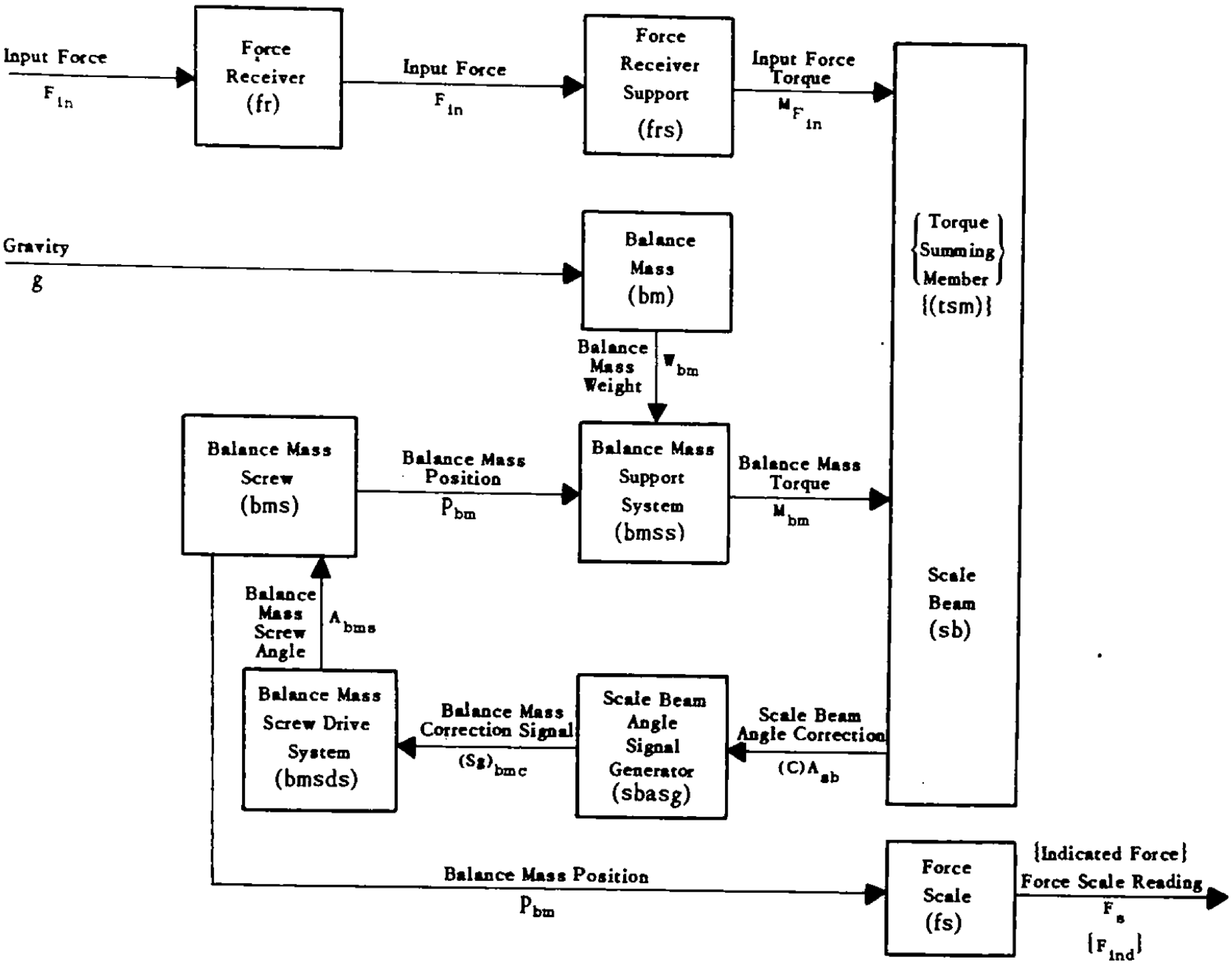

Fig. 1-9. Functional diagram of a force measuring scale with automatic balancing adjustment.

ADDITIONAL NOTATION CONVENTIONS

1.12. Several conventions in terminology and notation not previously mentioned are illustrated in Figs. 1-6 to 1-9. For example, in Fig. 1-8 dotted-line boxes enclosing several components are used to designate the scale beam angle signal generator and the balance mass screw drive system as subsystems. These subsystems are shown as single operating components in the functional diagram of Fig. 1-9. This convention of designating the components that may be conveniently grouped into a subsystem by a dotted-line enclosure is useful in many problems of instrument engineering.

1.13. In Fig. 1-9, the input to the signal generator is shown as the scale beam angle correction. This name implies that the designated angle is the angle through which the beam would have to be turned in order to bring it to the correct position. For the case of Fig. 1-9, the correct scale beam angle is zero, so that the correction angle is equal to the negative of the scale beam angle. The symbol used for the scale beam angle correction is a capital C, for correction, enclosed in parentheses, followed by the symbol for the scale beam angle. Another new notational convention is the use of the letters Sg enclosed in parentheses to denote "signal" and given subscript letters identifying the entity with which the signal is associated. The other symbols in Figs. 1-6 to 1-9 are formed in accordance with the conventions already described.

1.14. In the pictorial drawings of Figs. 1-6 and 1-8, the representation of a linear scale-and-index combination for producing output readings is considered to be suggestive only; the actual scale reading may be generated by counters, by printing recorders, or by any other means without violating the intended meaning of the diagram. A number of other similar conventions will appear in later discussions.

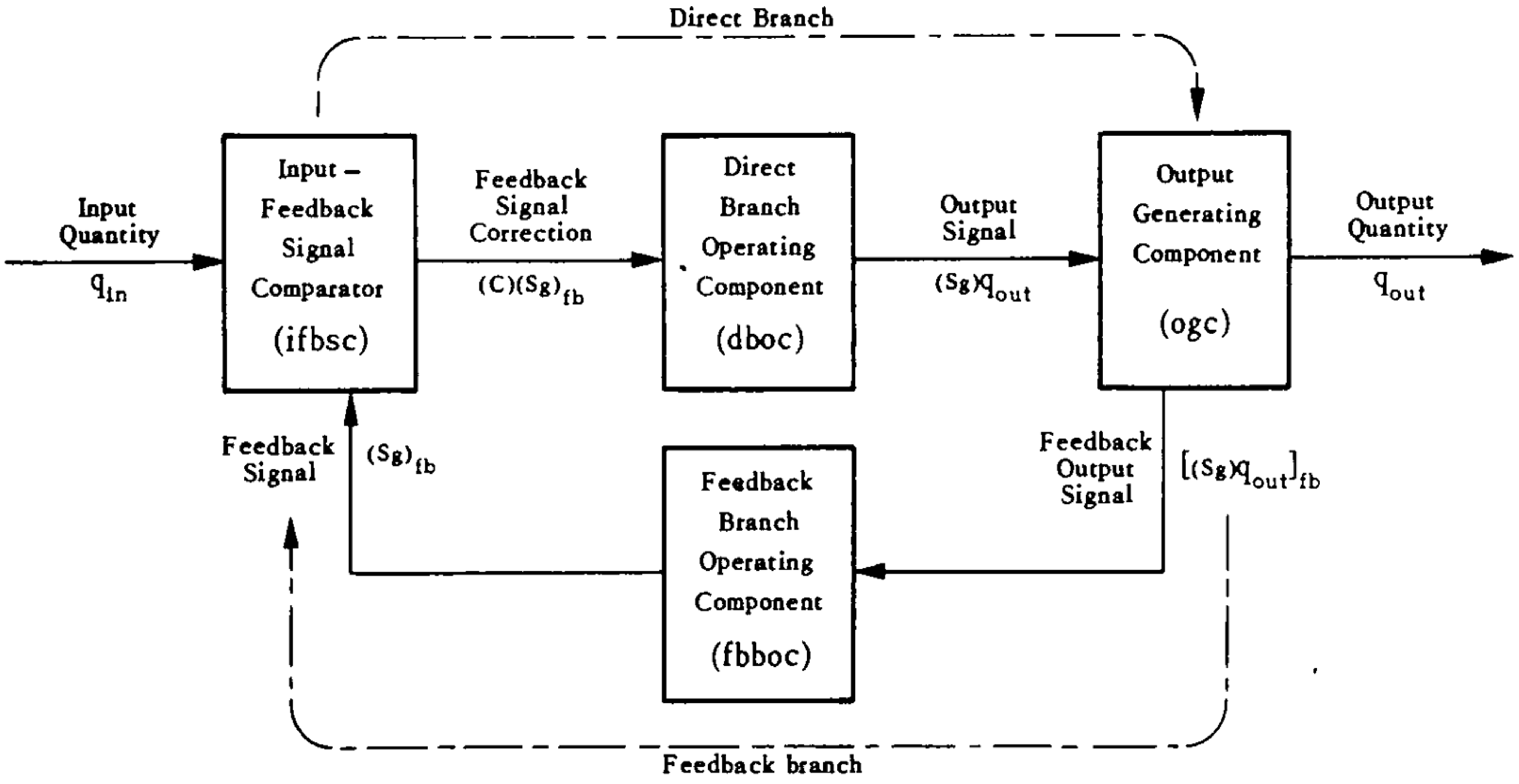

Fig. 1-10. Generalized functional diagram for a closed-chain system with a single feedback branch.

CLOSED-CHAIN SYSTEM WITH SINGLE FEEDBACK BRANCH

1.15. Figure 1-10 is a generalized functional diagram for a closed-chain system with a single feedback branch. The system input is received by the input-feedback signal comparator, which also produces the feedback signal correction as its output. The signal correction is the change in the feedback signal required to make it equivalent to the system input. The feedback signal correction is the input for the direct branch operating component, which produces the output signal, which in turn acts as the command quantity for the system output. The output signal is received by the output generating component and produces the system output as its own output. The output generating component has the feedback output signal as its second output. This signal is fed in the inverse direction as the input for the feedback branch operating component, which produces the feedback signal as its output.

AUTOMATIC FORCE MEASURING SCALE AS A CLOSED-CHAIN SYSTEM

1.16. The self-balancing scale arrangement of Figs. 1-8 and 1-9 is a special case of the generalized closed-chain system primarily because it continuously searches out the level of an unchanging input quantity. In order to be fitted into the generalized closed-chain pattern of Fig. 1-10, the automatic balancing system of Figs. 1-8 and 1-9 must be considered as rearranged to some extent. It is convenient to consider the zero signal generator output position of the scale beam as the unchanging input quantity and the actual position of the scale beam as the feedback quantity. The scale beam angle signal generator corresponds to the input-feedback signal comparator, and its output becomes the feedback signal correction. The balance mass screw drive system, the balance mass, and the balance mass support act as the direct branch operating component, while the scale beam becomes the feedback branch operating component. In Figs. 1-8 and 1-9 the output quantity is the indicated force, which is taken from the balance mass position by means of the scale. This arrangement does not fit in directly with the organization of components in Fig. 1-10 in that the feedback output signal is not taken from the output generating component but is drawn from an internal part of the direct branch operating component. However, because of the rigid mechanical coupling involved, the action of the illustrative system of Figs. 1-8 and 1-9 is

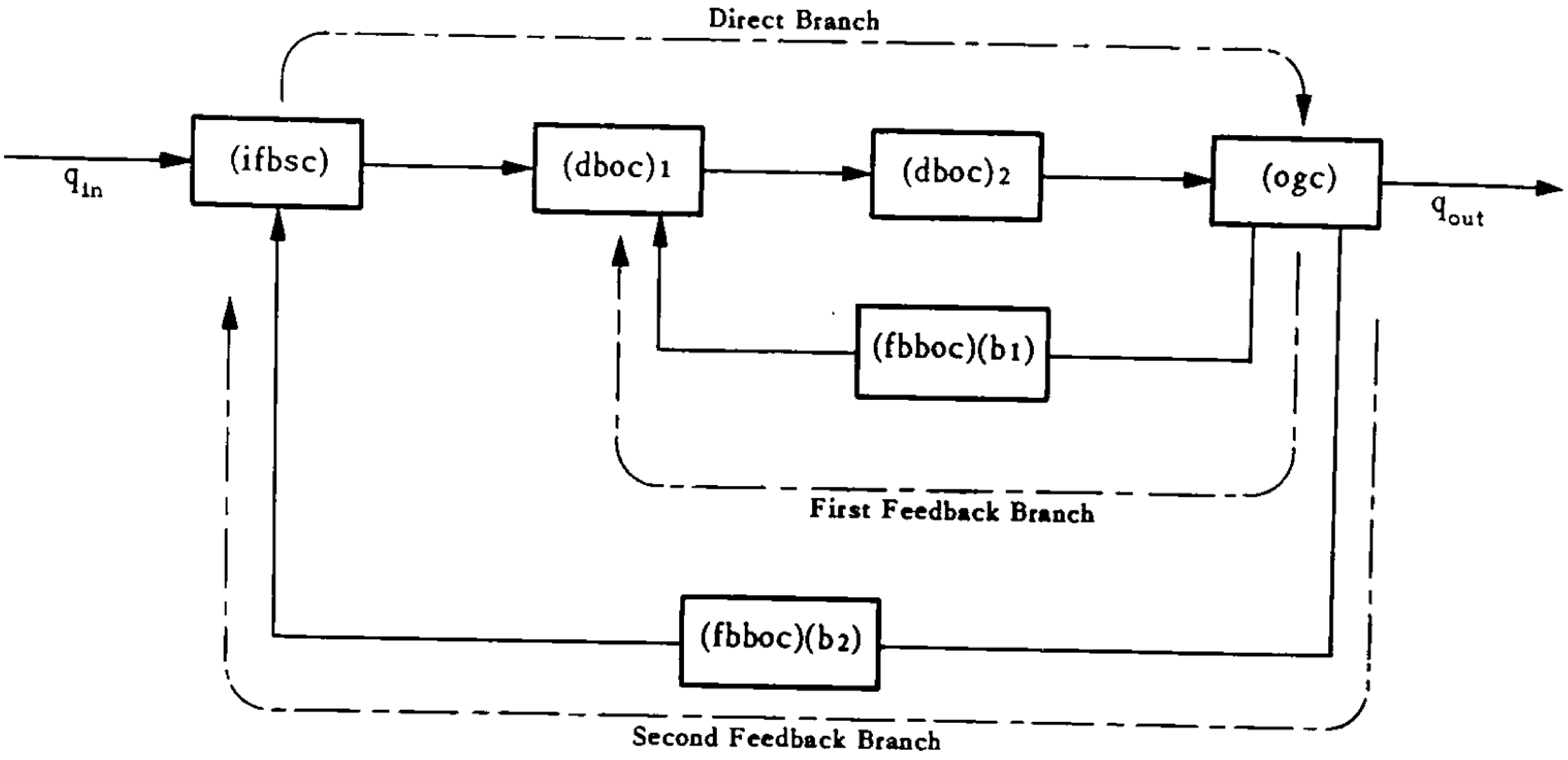

Fig. 1-11. Generalized functional diagram for a closed-chain system with multiple feedback branches.

clearly equivalent to that suggested by the diagram of Fig. 1-10. The departure from exact correspondence between the generalized diagram and the particular example cited is no worse than situations that often occur when actual applications of the general principles are considered.

CLOSED-CHAIN SYSTEMS WITH MULTIPLE FEEDBACK BRANCHES

1.17. Closed-chain systems are not restricted to single feedback branches, but may have an indefinite number of such branches interconnected with the primary direct branch in any one of many ways. Figure 1-11 illustrates a typical example of a multiple feedback branch closed-chain system. The generalized diagram is drawn with two feedback branches connected to two separate operating components earlier in the direct branch than the component from which the inputs to the feedback branches are taken. In order to reduce the size of the diagram while still clearly showing the functional connections, all quantity and component names are omitted from Fig. 1-11. These names are replaced by letter combinations formed by conventions that are generally similar to those used in Fig. 1-10. The exceptions that occur to the conventions already discussed are the addition of numbers and letters to identify the positions of components in direct or feedback chains. The chain branches themselves are identified by thin, curved dot-dash lines drawn so as to suggest the parts of the loop to which the branches belong.

CHAPTER 2

DIAGRAMS FOR DESCRIBING OPERATING SYSTEMS

GENERAL REQUIREMENTS

2.1. Conventions of various kinds have been introduced in the preceding chapter for the description of operating systems. However, the concepts have been restricted to the ideas required to introduce general principles, and did not include a number of diagram types that are generally useful. A more extended discussion of diagrams and their applications is given in this chapter.

2.2. The effectiveness of engineering work on an instrument system is greatly improved when the system functions are clearly described by means of diagrams and mathematical forms. To be completely satisfactory the presentation must not only include descriptions of mechanical arrangements in terms of pictures or pictorial diagrams, but must also associate this pictorial data with functional and mathematical representations. The latter two diagram types consist of simple blocks properly labeled to represent the actions of components represented to individuals already generally familiar with the system being described. In practice, the design of any given system must be started by engineers who have such broad backgrounds of knowledge that all the necessary information can usually be presented by means of simple diagrams. On the other hand, a more extensive set of diagrams is required to explain a complete system to interested persons who have a limited experience with similar arrangements. Figures 2-1, 2-2, 2-3, 2-4, 2-5, and 2-6 form an illustrative series of diagrams of this kind for a diaphragm-type pressure gage.

CUTAWAY VIEW

2.3. Figure 2-1 is a cutaway view of the pressure gage,[*] and is used to illustrate a typical mechanical detail picture. The diaphragm unit which receives the pressure to be measured, the coupler, the pressure index, and the scale are labeled in Fig. 2-1 as components with essential functions in the operation of the gage. Cutaway views like that of Fig. 2-1 are intended to show the parts making up a system, the mechanical forms of these parts, and how the parts are physically related. The object of a diagram of this type is to describe the arrangement for individuals who wish to familiarize themselves with a system under discussion. No attempt is made in diagrams of this kind to indicate functional relationships beyond those directly evident from the mechanical features of the picture.

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[*] The unit shown is actually the indicating head of a pitot-static airspeed meter for aircraft. Because the example used here considers the unit as a pressure gage only, no reference is made at this place in the text to its role as part of an airspeed meter system.

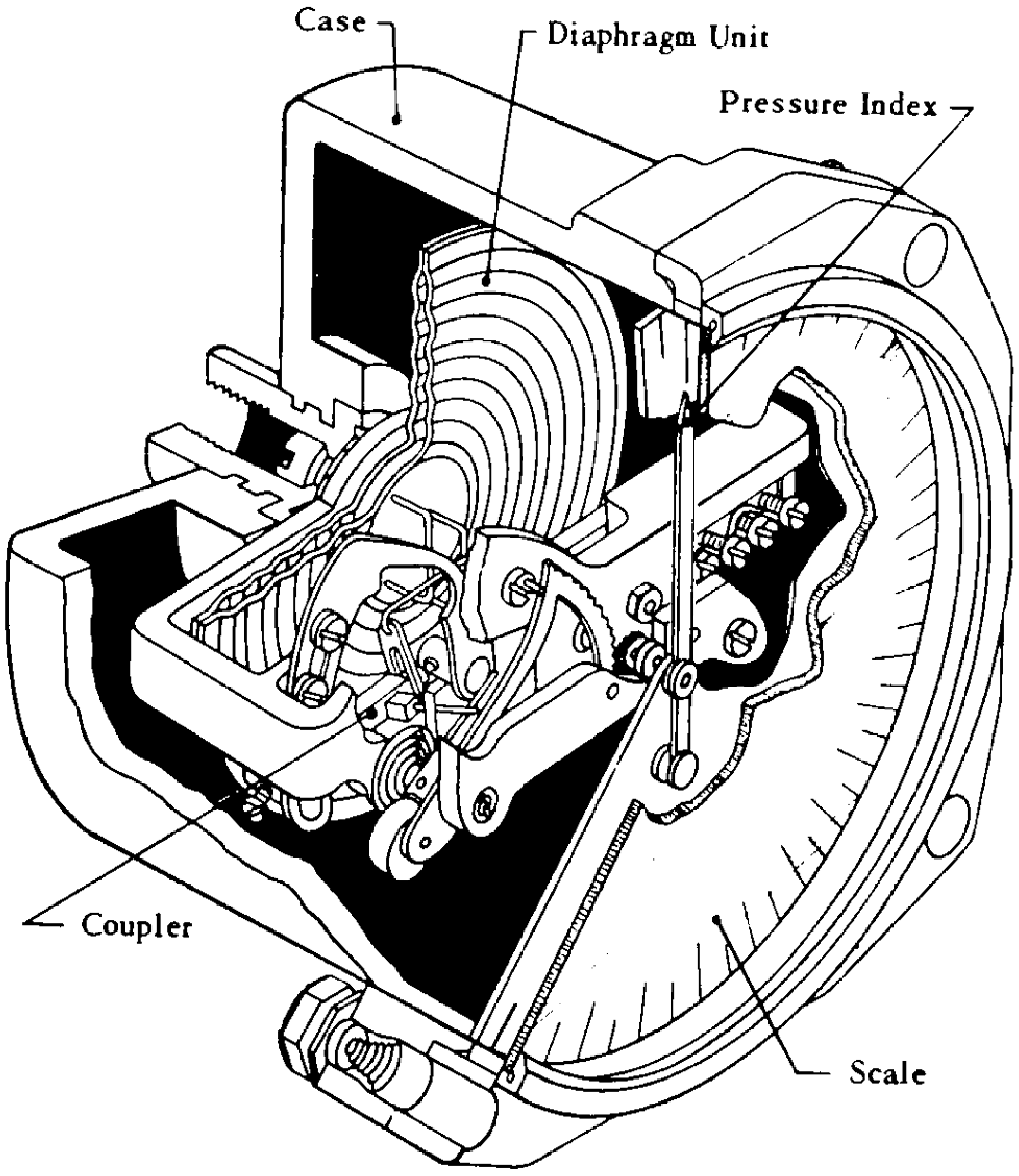

Fig. 2-1. Cutaway view of diaphragm-type pressure indicator: mechanical detail picture.

PICTORIAL SCHEMATIC DIAGRAM

2.4. Functional relationships among the operating components of a system may be shown in diagrams of many kinds, ranging from pictorial diagrams that obviously represent parts of a mechanical feature picture, to mathematical diagrams made up of line boxes that are either empty except for simple identification* symbols, or contain only mathematical expressions.** Pictorial schematic diagrams greatly facilitate discussions of operation among individuals having various levels of knowledge about the system being considered; however, they are difficult to draw and are not necessary for engineers who are completely familiar with the functional relationships involved. Figure 2-2 is an illustrative pictorial diagram showing the essential operating components for the pressure gage of Fig. 2-1, with the components placed so that their functional relationships are directly apparent.

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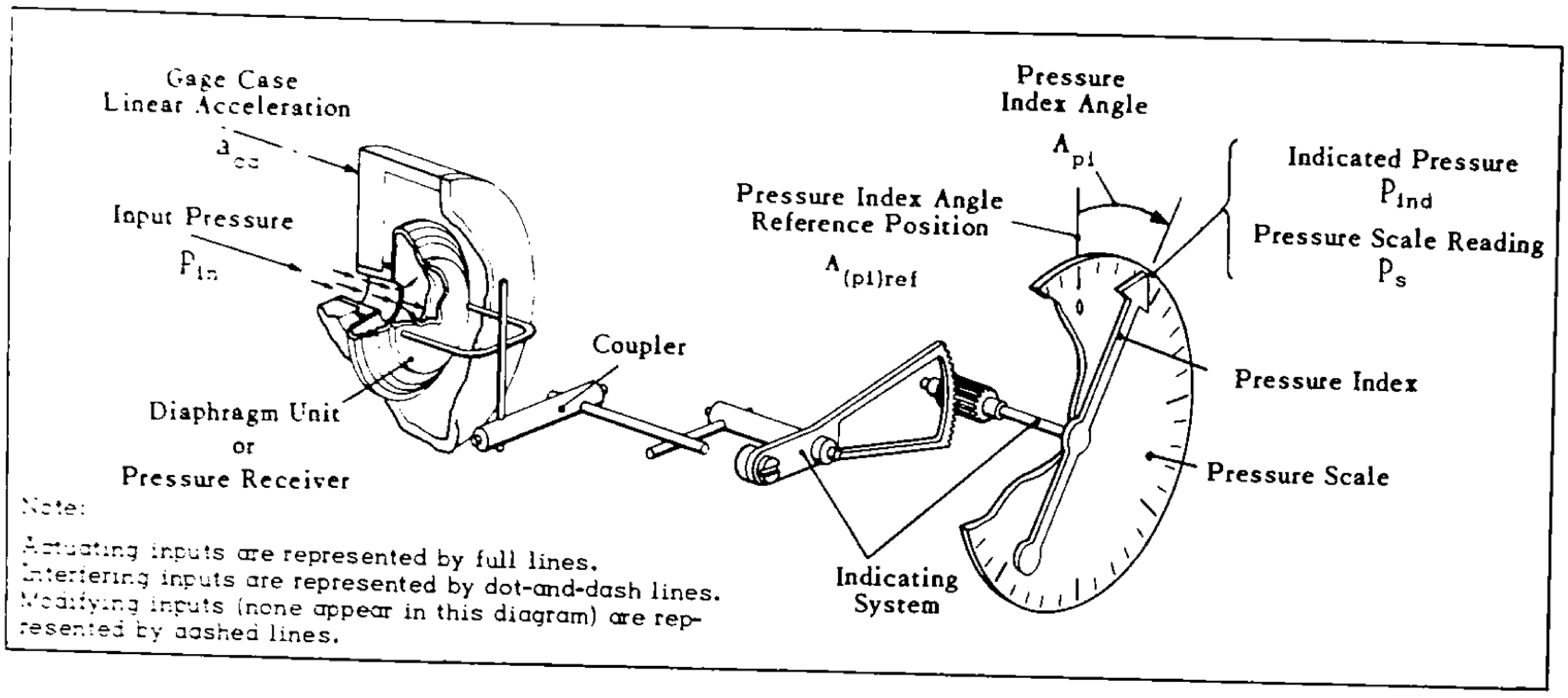
* This identification may be by letters, by numbers, or by a combination of letters and numbers.

** It is feasible to place mathematical symbols in the boxes of a diagram only when the functions to be represented are relatively simple. When complicated functions must be used, it is better to set up a performance characteristic table with the necessary association established by proper identification symbols.



Note:

Actuating inputs are represented by full lines.
Interfering inputs are represented by dot-and-dash lines.
Modifying inputs (none appear in this diagram) are represented by dashed lines.

Fig. 2-2. Illustrative pictorial schematic diagram for a diaphragm-type pressure gage.

INPUT CATEGORIES FOR THE OPERATING SYSTEM

2.5. Any independent quantity which is able to affect the output of an operating system is properly one of the inputs for the system. In practice, inputs naturally divide into three categories:

1) Actuating inputs, which are the quantities intended to act as the primary independent quantities associated with the operating component.
2) Interfering inputs, which have undesirable effects on the operating component output and in practice are minimized so far as possible by design.
3) Modifying inputs, which act on the operating component to alter relationships between the actuating inputs and the output quantity.

2.6. The illustrative pictorial schematic diagram of Fig. 2-2 shows that the actuating input is pressure, and shows the linear acceleration of the gage case as an interfering input. The general convention which is illustrated in the diagram shows that the distinction between inputs of these two types is made by using full lines to represent actuating inputs and dash-dot lines to represent interfering inputs.

FUNCTIONAL IDENTIFICATION OF OPERATING SYSTEMS

2.7. Functions of three types are usually associated with the action of any operating system:

1) It must receive input quantities, and for this reason may reasonably be classed as a receiver.

2) It must act as a transducer,[*] in the sense that it receives power from one system and supplies power to another system; this action may also be recognized by calling the operating component a coupler.

3) It must act as a generator, in the sense that it generates an output of some kind.

2.8. In practice, the name applied to any given operating component depends upon which of its three phases of behavior appears to be of primary importance in a particular application. For example, an electronic unit designed to be actuated by electromagnetic waves is generally called a radio receiver because its important action is that of responding to its actuating input quantity. The loud speaker associated with a radio receiver is a transducer because it functions by taking electrical power from the receiving system and supplying acoustic power to the air in contact with the speaker diaphragm. A conventional dynamo receives mechanical power and generates electrical power; because in this example attention is primarily on the output of the unit, the machine concerned is a generator. The principles outlined here are applied in forming the names used for the operating components of Fig. 2-1.

2.9. In Fig. 2-2, the diaphragm unit at the left of the diagram is the pressure receiver, which is driven by the input pressure to produce a linear displacement as its output. This linear displacement is received by the coupler, which is a lever arrangement that supplies a linear displacement as the input for the indicating system. A gear-and-pinion arrangement acts as the indicating system, which receives the linear displacement output of the coupler and produces the pressure index angle. This quantity is the angular displacement of the pressure index from some reference orientation determined by the position occupied by the index when no pressure is applied to the gage. The zero point of the pressure scale is made to coincide with the zero pressure position of the index.[**] The scale is calibrated so that numbers on its face are measures of the input pressure applied to the gage. With a given pressure input, the number corresponding to the index position is the pressure scale reading, or the indicated pressure, which is the essential output from the gage. With a perfect pressure gage, the scale reading would be equal to the input pressure under all conditions. An ideal pressure gage would give scale readings equal to the input pressure except for certain errors tolerated in the design. The readings from any actual pressure gage will differ from the readings of the ideal pressure gage by the presence of uncertainties. These distinctions will not appear explicitly on diagrams like that of Fig. 2-2, which in general may be descriptive of any pressure gage with the features shown.

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[*] Webster's Collegiate Dictionary gives for transducer: a device actuated by power from one system and supplying power to a second system.

[**] In practice, measuring instruments are often built with suppressed zero scales. In instruments of this type, only a portion of the range of input levels appears on the scale itself, and stops within the mechanism restrict the index position to motion over this region.

PICTORIAL-FUNCTIONAL DIAGRAM

2.10. Figure 2-3 is an illustrative pictorial-functional diagram for the pressure gage of Fig. 2-2. Functional information is combined with the pictorial representation by showing the mechanism as divided into operating components enclosed in a chain of dashed-line boxes. Enough details of the arrangement for carrying out the described function are included for the identification of parts on the pictorial-functional diagram.

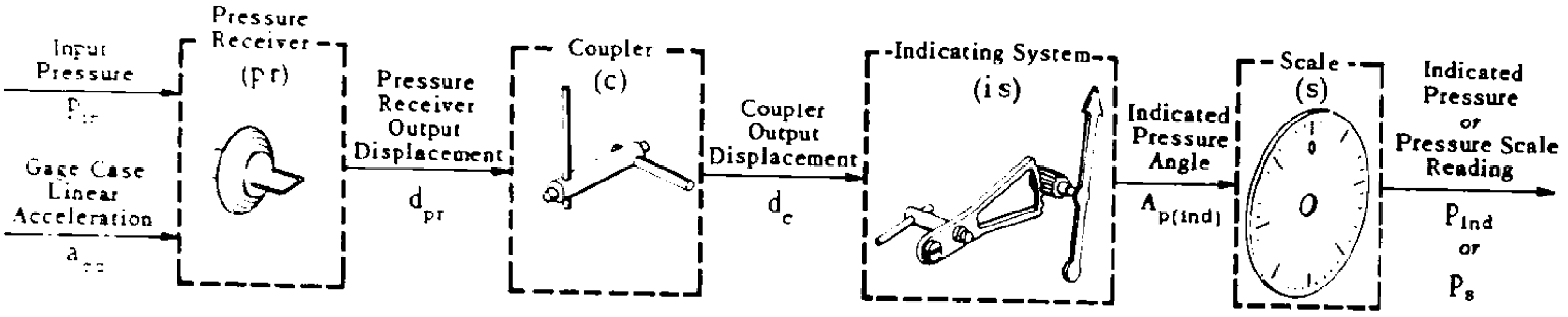

Fig. 2-3. Illustrative pictorial-functional diagram for a diaphragm-type pressure gage.

NAMES AND SYMBOLS ASSOCIATED WITH THE PRESSURE GAGE

2.11. In order for a diagram like that of Fig. 2-3 to be useful in practice, the physical quantities and components involved must be given names and symbols. For example, the input pressure, p_{in}, (p for pressure, and the subscript in for input) and the linear acceleration of the gage case, a_{ca}, (a for linear acceleration, ca for case) are shown as inputs to the pressure receiver, (pr), (p for pressure, r for receiver). Pressure as the actuating input is represented by a solid-line arrow, showing that the pressure level is an independent variable determined by some component or system to the left of the pressure receiver in the direct chain of pressure gage operating components. The fact that the pressure receiver responds to linear acceleration* is due to the necessity of using materials with mass and elasticity in its construction, and is incidental to the action of the unit in responding to pressure. The dot-dash-line arrow identifies this quantity as an interfering input. Linear acceleration falls into the class of interfering inputs to the pressure gage because its effect on the index position is undesirable and in this sense interferes with proper operation of the gage.

2.12. The output from the pressure receiver is the pressure receiver output displacement, d_{pr}, (d for linear displacement, and pr as a subscript showing that the displacement is associated with the pressure receiver). The coupler (symbol (c)) receives the pressure receiver displacement and produces the coupler output displacement (symbol d_c). The coupler output displacement acts as input for the indicating system, (symbol (is)), which in turn produces the indicated pressure angle, $A_{p(ind)}$, (A for angle, and (ind) for indicated) as its output. The indicated pressure angle, read with respect to the scale (symbol (s)), gives the pressure scale reading, p_s, (p for pressure, and subscript s for scale), which is also called the indicated pressure, p_{ind}, (p for pressure, and subscript ind for indicated), because it is the pressure indicated by the gage.

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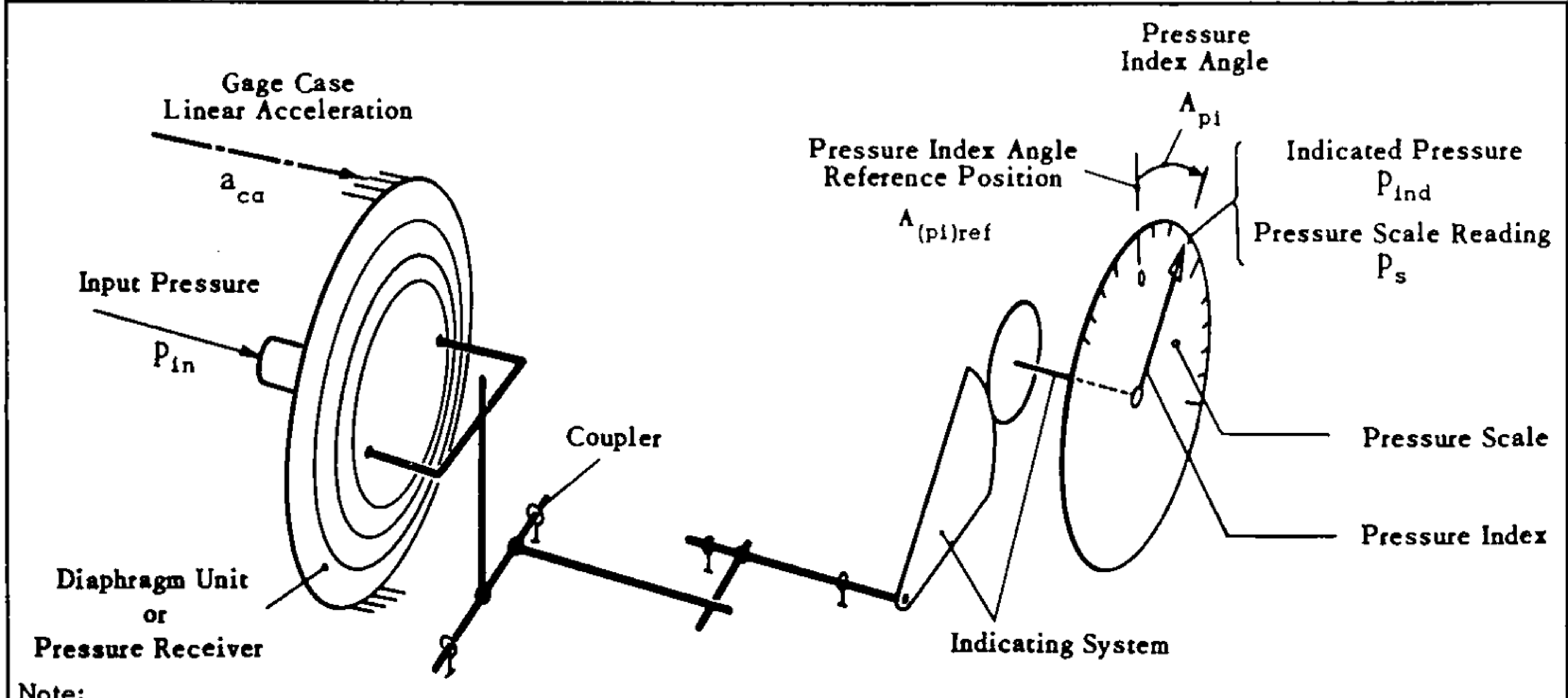
* Modern pressure gages are usually designed to minimize this effect by having properly located support points to reduce the effect of acceleration forces.

2.13. The pressure gage used in Figs. 2-1, 2-2, and 2-3 to illustrate three generally useful methods for describing operating systems is a measuring instrument in the sense that its essential output is a measurement of the pressure that is its actuating input.

INDICATIONS AND SCALE READINGS

2.14. In practice, a mechanism similar to that described in the figures may not be required to produce scale readings, but only to generate indications in terms of index positions, without any numerical results appearing at any time. In situations of this kind the general methods for description would be identical with those illustrated in Figs. 2-1, 2-2, and 2-3, but the scale would be omitted from the chain of components. Ordinarily this chain would be lengthened by the addition of new components designed to accept the pressure index angle as one of its inputs. The more complicated systems that result from lengthened operating chains are discussed later.



Note:
Actuating inputs are represented by full lines.
Interfering inputs are represented by dot-and-dash lines.
Modifying inputs (none appear in this diagram) are represented by dashed lines.

Fig. 2-4. Line schematic diagram for a diaphragm-type pressure gage.

LINE SCHEMATIC-FUNCTIONAL DIAGRAM

2.15. Because of the drawing skill required for making pictorial drawings, diagrams of this type are often replaced by line schematic representations, which may be made more easily, and, when they are properly understood, serve the same purpose as the more complicated presentations. Figures 2-4 and 2-5 are diagrams that effectively convey the same information as the pictorials of Figs. 2-2 and 2-3. All names and symbols are identical for the two sets of diagrams, so the previous discussion of the earlier figures applies equally well to Figs. 2-4 and 2-5.

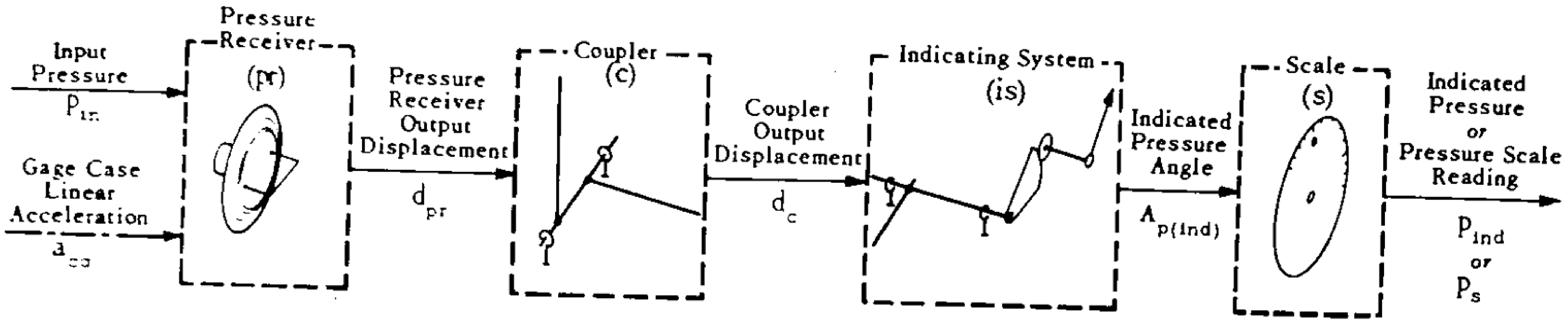


Fig. 2-5. Line schematic-functional diagram for a diaphragm-type pressure gage.

FUNCTIONAL DIAGRAM

2.16. Pictorial- and line schematic-functional diagrams are useful for the purpose of associating functions with the physical forms of operating components, but are unnecessarily complicated when this association is clearly understood. If this situation exists, functional diagrams are easily made with straight-line boxes and may be made to supply the same information as pictorial-functional diagrams. Figure 2-6 is a functional diagram for the illustrative pressure gage that corresponds to the more elaborate diagrams of Figs. 2-3 and 2-5.

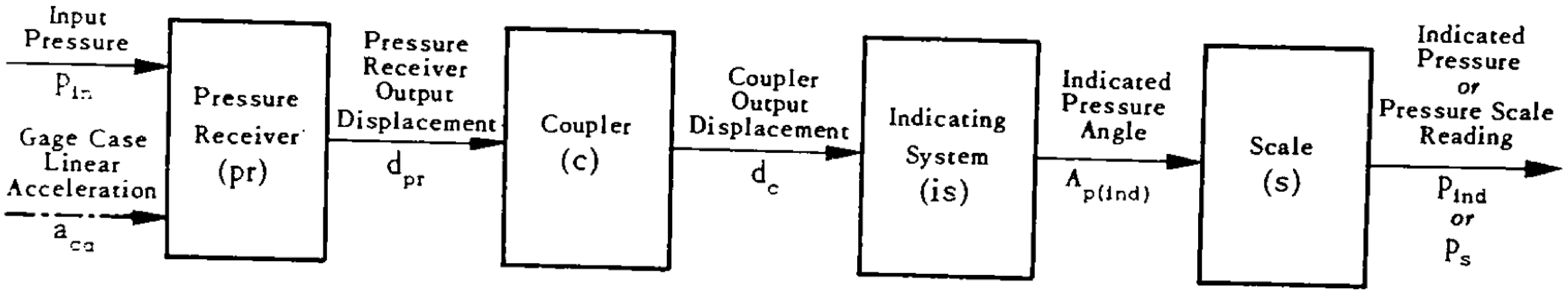


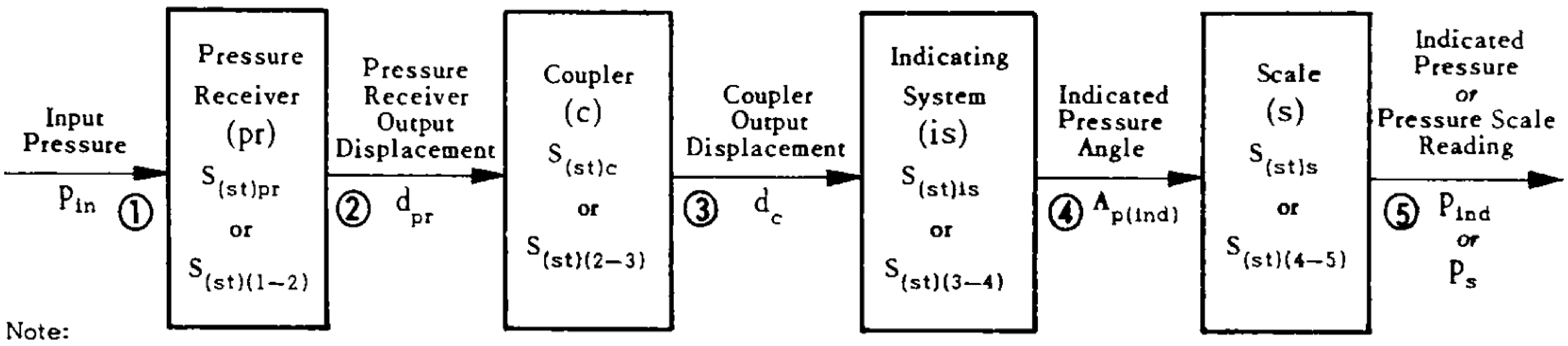
Fig. 2-6. Illustrative functional diagram for a diaphragm-type pressure gage.

UTILITY OF SEVERAL DIAGRAMMATIC FORMS

2.17. Input and output quantity names and symbols are identical in the diagrams of Figs. 2-2 to 2-6 inclusive. The same name is used for each component in all these diagrams, which are equivalent for any person familiar with the details of the implementation involved. A comparison of the diagrams makes it immediately obvious that if the pressure gage construction is not already familiar, the diagrams of Figs. 2-3, 2-4, and 2-5 convey more information than the diagram of Fig. 2-6. In any practical situation the type of diagram to be used depends upon circumstances. It is often worth while for reference purposes to make up a series of diagrams more or less similar to those of Figs. 2-1 to 2-6.* With this series available, the needs of a particular situation may be met by selecting diagrams appropriate to the background knowledge of the individuals concerned in any given discussion.

MATHEMATICAL DIAGRAM

2.18. When the theoretical aspects of system or component operation are of primary interest, the diagrams already discussed may be replaced by the mathematical diagram. In this diagram type, straight-line boxes similar to those of the functional diagram are used to represent operating components or combinations of operating components. Each box contains a mathematical expression that describes the performance characteristics of the component or system represented by the box. In order to facilitate associations between physical units and the corresponding mathematical performance descriptions, the names and symbols for the components concerned may be included in the boxes. The performance-describing mathematical expressions may be identified either with the components whose behavior they represent, or with the interconnections between operating components. For example, in the illustrative mathematical diagram of Fig. 2-7, which shows the behavior of each component in the pressure gage of Figs. 2-1 to 2-6, the performance symbol for the pressure receiver is identified either by the subscript symbol (pr) (for pressure receiver) or by the subscript (1-2)** to indicate


Input Pressure
P_{in} ①
Pressure Receiver (pr)
$S_{(st)pr}$ or $S_{(st)(1-2)}$
Pressure Receiver Output Displacement
② d_{pr}
Coupler (c)
$S_{(st)c}$ or $S_{(st)(2-3)}$
Coupler Output Displacement
③ d_c
Indicating System (is)
$S_{(st)is}$ or $S_{(st)(3-4)}$
Indicated Pressure Angle
④ $A_{p(ind)}$
Scale (s)
$S_{(st)s}$ or $S_{(st)(4-5)}$
Indicated Pressure or Pressure Scale Reading
⑤ P_{ind} or P_s

Note:

To simplify the diagram only static performance symbols, represented by the sensitivity, S_{st}, which is by definition equal to the ratio between static output changes and the corresponding static input changes, are given.

The unhyphenated letter combination subscripts identify the performance symbol with the operating component which it represents.

The hyphenated number subscripts identify the performance symbol with the action between the points of interconnection of the components of the operating chain.

When performance characteristics are of primary interest, identifying names and symbols may be omitted, leaving only performance symbols in the operating component boxes.

The performance symbols of this figure are incomplete in that input-output subscripts are omitted. Rules for forming complete symbols are given in Chapter 10.

Fig. 2-7. Illustrative mathematical diagram for a diaphragm-type pressure gage.

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* When the system to be represented involves electrical and magnetic components, conventional circuit diagrams will be included in the descriptive series as a matter of course. The pressure gage used in the illustrative example does not include any electrical components, so the series of diagrams does not contain a circuit diagram.

** The hyphen between the diagram point-identifying numbers is conventionally used to prevent misunderstanding when the numbers have more than one digit each. When it is apparent that the identifying numbers have only one digit, the hyphen may be omitted.

that the performance symbol describes action occurring between diagram point 1 and diagram point 2.

2.19. In the illustrative mathematical diagram of Fig. 2-7, the simplest possible type of performance is postulated. The corresponding performance-describing concept is static sensitivity, which is by definition the ratio of a small change in the output quantity between constant levels to the corresponding small change in the input quantity between constant levels. The performance symbol for static sensitivity uses a capital S (for sensitivity) as its main symbol and the subscript st (for static). Other subscript symbols are added as they become necessary for identification purposes. The conventions for forming complete subscripts are considered in Chapter 10, which describes methods for representing performance-describing concepts that are more generalized than the concept of static sensitivity.

SYMBOL DIAGRAM

2.20. Discussions among engineers already thoroughly familiar with a problem are facilitated by the use of symbol diagrams similar to that of Fig. 2-8 for the diaphragm-type pressure gage. This diagram is made up of small boxes connected by arrows and identified by the letter symbols for the operating components of Figs. 2-6 and 2-7. The system input and the system output are represented by symbols, but all identification of intermediate physical quantities is eliminated. The advantage of the symbol diagram is that it is easy to draw, it is small, and, for many purposes of the instrument engineer, it is as effective as more complicated diagrams. In practice, symbol diagrams are often accompanied by performance information tables (symbol PIT), which summarize all the data required to specify the actions associated with the system described by the diagram. Many examples that include tables of this kind appear in later chapters.

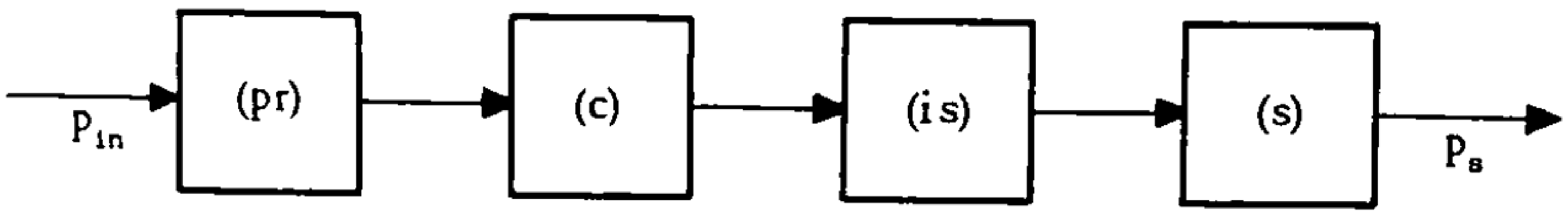


Fig. 2-8. Illustrative symbol diagram for a diaphragm-type pressure gage

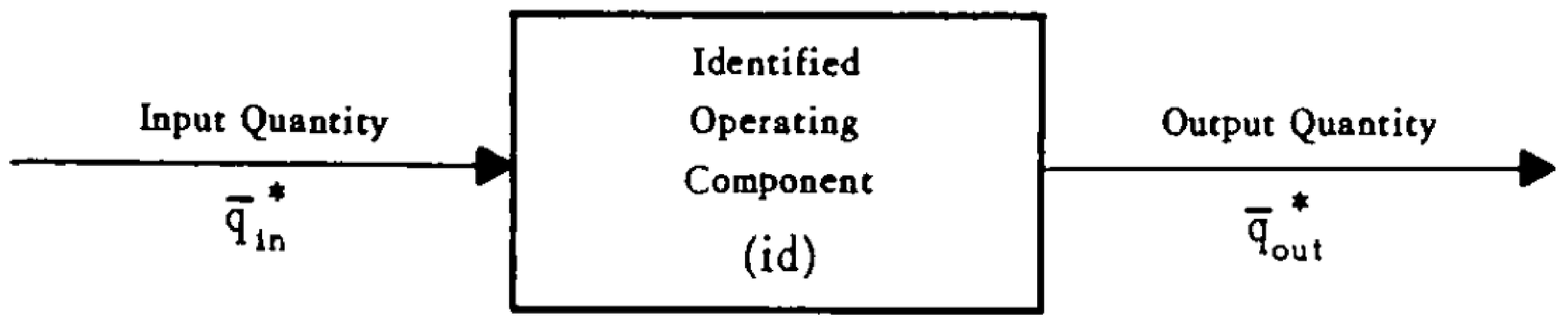
CHAPTER 3

PERFORMANCE OPERATORS AND PERFORMANCE FUNCTIONS FOR OPERATING SYSTEMS

PERFORMANCE OPERATORS

3.1. Performance-describing concepts are of primary importance for instrument engineering. These concepts are used to specify the behavior of operating components in causing output quantities to be functions of input quantities. Many variations of this basic idea are required to meet the needs of particular situations.

3.2. The most comprehensive performance-describing concept is the vector performance operator, illustrated in Fig. 3-1 for a single operating component



a) Typical identified operating component

$$\bar{q}_{out} = [\overline{PO}]_{id}\,\bar{q}_{in}$$

$[\overline{PO}]_{id}$ = Vector Performance Operator for the Identified Operating Component.

b) Vector performance operator

* Bars over the symbols show that the quantities represented are vectors.

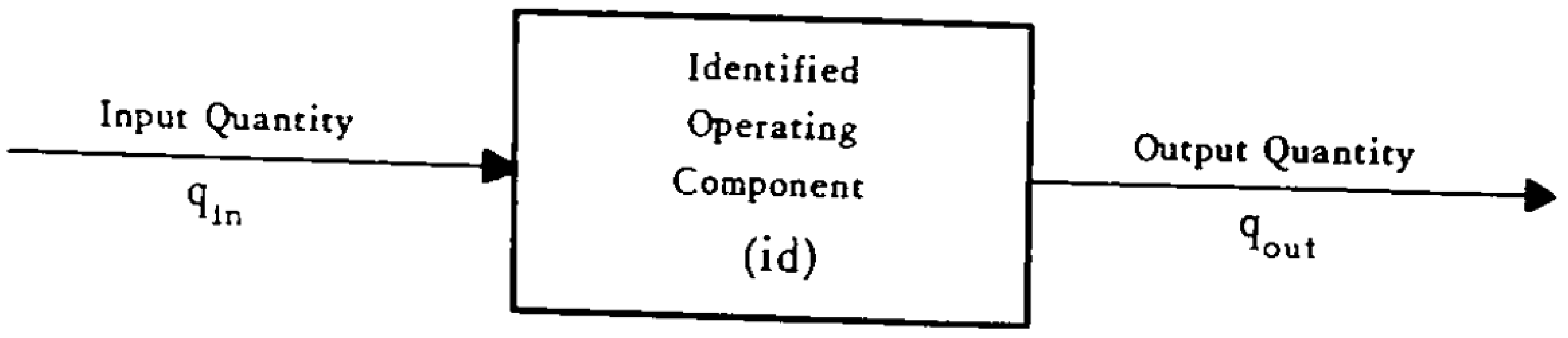
Fig. 3-1. The vector performance operator.

The main symbol for this operator is formed by the capital letters PO, in brackets, and with a superscript bar $\overline{}$, to form $[\overline{PO}]$. Subscripts are added to this symbol as required to identify the operating component with which it is associated. In practice, any complete operator symbol will carry additional subscripts to identify the input and output quantities considered. These subscripts are unnecessary for present purposes and are omitted in order to simplify the symbols.

3.3. The formal performance equation of Fig. 3-1 shows that the vector output quantity is equal to the result of the identified performance operator acting on the vector input. This equation concisely summarizes the meaning of the vector performance operator insofar as it is associated with the behavior of an operating component with vector properties. By definition, these operators describe not only magnitude changes but also response delay effects and geometrical actions in space. In general, they may have forms that depend upon mathematical expressions, tables, curves, or any other means for conveying information on performance.

SCALAR PERFORMANCE OPERATORS

3.4. The scalar performance operator represented in Fig. 3-2 is similar in all respects to the vector performance operator except that the operating component involved has only scalar action on a scalar input to produce a scalar output. The symbol for the scalar performance operator is similar to that for the vector performance operator with the superscript bar omitted. A special form of the performance operator is the performance function, whose properties are discussed in the next paragraph.



a) Typical identified operating component

$q_{out} = [PO]_{id} q_{in}$

$[PO]_{id}$ = Scalar Performance Operator for the Identified Operating Component

b) Scalar performance operator

$q_{out} = [PF]_{id} q_{in}$

$[PF]_{id}$ = Performance Function for the Identified Operating Component

c) Performance function

Fig. 3-2. The scalar performance operator and the performance function.

PERFORMANCE OPERATORS FOR OPEN-CHAIN SYSTEMS

3.5. The action of an open chain of operating components may be represented by means of a single performance operator that is formed from the performance operators of the individual operating components. The performance operator for a given single component applied to the input of this component gives the corresponding output. Each component in the chain receives the output of the preceding component as its input and produces an output which is related to the input by the performance operator of the component. The formal process involved is illustrated in symbols by Note 5 of Fig. 3-3. In general, the output from any chain of components can be expressed by successive applications of the corresponding performance operators to the result obtained by applying the operator of the first component to the system input. When the operators have space vector* properties, or the components have nonlinear characteristics, this process of applying operators must be carried out with careful attention to the order of the components in the chain. In many situations performance operators do not have vector properties, but do have mathematical properties that make it possible for them to be combined into groupings without regard to order. If only scalar relationships are involved, and all the operating components are describable by linear mathematical expressions, the performance operators may often be combined in any order. When performance operators have this property, they reduce to performance functions. Figure 3-3 shows the grouping of individual performance operators to give the resultant operators for the subsystems and the system shown. All operators are identified in terms of both letters and numbers for operating components or systems, and also in terms of the numbers for the chain connection points between which the action described by the operators occurs. The subscript parenthesis which associates an operator with chain connection points has the identification of the first point nearest the main symbol and the identification of the second point to the right of and separated from the first point identification by a hyphen. In forming the resultant operator for a chain of components, the individual operators are written in the order of the components whose action they represent. It is always necessary to recognize any requirements of order in the application of individual operators when the formal expressions are used in practice.

---

* Vector properties for the performance operator are shown formally by placing a bar over the operator symbol. Because vector properties are of only incidental importance here, the bar is omitted from the symbols of Fig. 3-3.

PERFORMANCE OPERATORS FOR CLOSED-CHAIN SYSTEMS

3.6. Performance operators for components grouped in closed-chain configurations are formed by a procedure identical with that already explained for open-chain systems in the discussion of Fig. 3-3. Figure 3-4 is a functional-mathematical diagram for a generalized closed-chain system in which all the operating components have characteristics that may be represented by performance functions. Identifying subscripts to associate the performance functions with particular operating connections and chain connection points are formed under the conventions already described for Fig. 3-3. The physical quantities that serve as inputs and outputs are identified with operating components and chain connection points by the subscript conventions of Fig. 3-3.

3.7. The system of Fig. 3-4 has an input receiver that responds to an input quantity from some source external to the system. The input receiver generates the input signal* as its output, which represents the input quantity and is the same kind of quantity as the feedback signal. The input signal and the feedback signal are both supplied as inputs to the input signal – feedback signal comparator.

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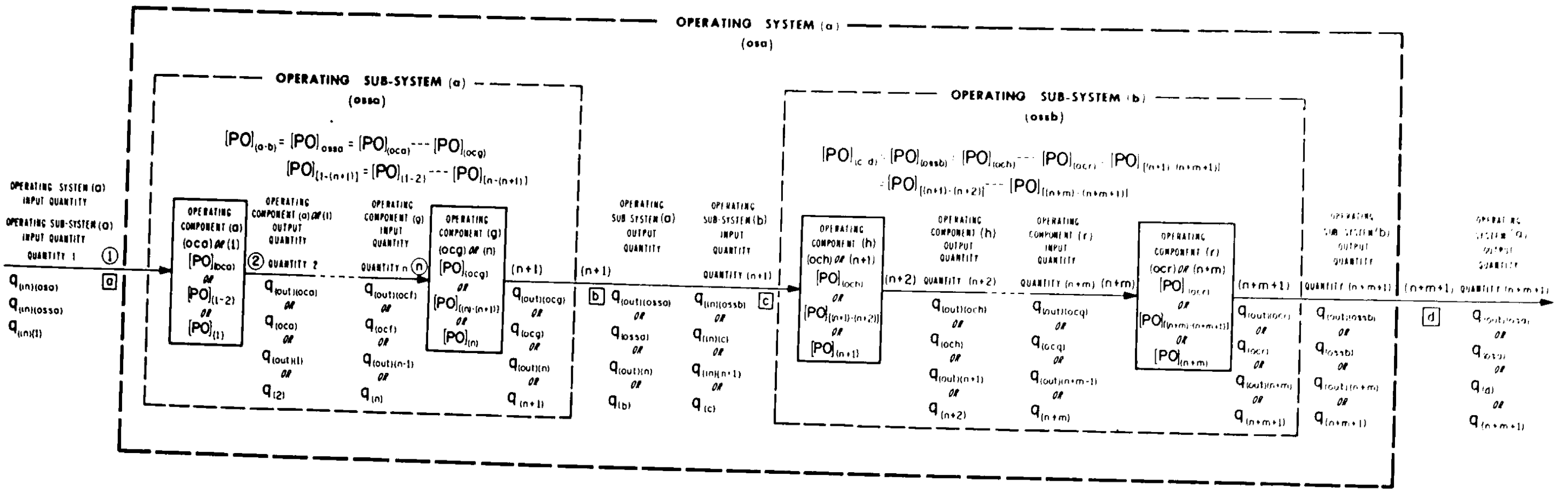
* The symbol for a signal representing a quantity is Sg in parentheses, followed by the symbol for the quantity that is represented by the signal. The symbol for a signal associated with an operating component for which it is the output is (Sg), with a subscript identifying the component but without any following symbol for a quantity.


OPERATING SYSTEM (a)
(osa)
OPERATING SUB-SYSTEM (a)
(ossa)
$[PO]_{(a-b)} = [PO]_{ossa} = [PO]_{(oca)} --- [PO]_{(ocg)}$
$[PO]_{[1-(n+1)]} = [PO]_{(1-2)} --- [PO]_{[n-(n+1)]}$
OPERATING SYSTEM (a) INPUT QUANTITY
OPERATING SUB-SYSTEM (a) INPUT QUANTITY
QUANTITY 1
①
a
$q_{(in)(osa)}$
$q_{(in)(ossa)}$
$q_{(in)(1)}$
OPERATING COMPONENT (a)
(oca) OR (1)
$[PO]_{(oca)}$ OR $[PO]_{(1-2)}$ OR $[PO]_{(1)}$
OPERATING COMPONENT (a) OR (1) OUTPUT QUANTITY
② QUANTITY 2
$q_{(out)(oca)}$ OR $q_{(oca)}$ OR $q_{(out)(1)}$ OR $q_{(2)}$
OPERATING COMPONENT (g) INPUT QUANTITY
QUANTITY n ⓝ
$q_{(out)(ocf)}$ OR $q_{(ocf)}$ OR $q_{(out)(n-1)}$ OR $q_{(n)}$
OPERATING COMPONENT (g)
(ocg) OR (n)
$[PO]_{(ocg)}$ OR $[PO]_{[(n)-(n+1)]}$ OR $[PO]_{(n)}$
(n+1)
$q_{(out)(ocg)}$ OR $q_{(ocg)}$ OR $q_{(out)(n)}$ OR $q_{(n+1)}$
OPERATING SUB SYSTEM (a) OUTPUT QUANTITY
(n+1)
b
$q_{(out)(ossa)}$ OR $q_{(ossa)}$ OR $q_{(out)(n)}$ OR $q_{(b)}$
OPERATING SUB-SYSTEM (b) INPUT QUANTITY
QUANTITY (n+1)
c
$q_{(in)(ossb)}$ OR $q_{(in)(c)}$ OR $q_{(in)(n+1)}$ OR $q_{(c)}$
OPERATING SUB-SYSTEM (b)
(ossb)
$[PO]_{(c-d)} = [PO]_{(ossb)} = [PO]_{(och)} --- [PO]_{(ocr)} = [PO]_{[(n+1)-(n+m+1)]}$
$= [PO]_{[(n+1)-(n+2)]} --- [PO]_{[(n+m)-(n+m+1)]}$
OPERATING COMPONENT (h)
(och) OR (n+1)
$[PO]_{(och)}$ OR $[PO]_{[(n+1)-(n+2)]}$ OR $[PO]_{(n+1)}$
(n+2)
OPERATING COMPONENT (h) OUTPUT QUANTITY
QUANTITY (n+2)
$q_{(out)(och)}$ OR $q_{(och)}$ OR $q_{(out)(n+1)}$ OR $q_{(n+2)}$
OPERATING COMPONENT (r) INPUT QUANTITY
QUANTITY (n+m)
(n+m)
$q_{(out)(ocq)}$ OR $q_{(ocq)}$ OR $q_{(out)(n+m-1)}$ OR $q_{(n+m)}$
OPERATING COMPONENT (r)
(ocr) OR (n+m)
$[PO]_{(ocr)}$ OR $[PO]_{[(n+m)-(n+m+1)]}$ OR $[PO]_{(n+m)}$
(n+m+1)
$q_{(out)(ocr)}$ OR $q_{(ocr)}$ OR $q_{(out)(n+m)}$ OR $q_{(n+m+1)}$
OPERATING SUB-SYSTEM (b) OUTPUT QUANTITY
QUANTITY (n+m+1)
$q_{(out)(ossb)}$ OR $q_{(ossb)}$ OR $q_{(out)(n+m)}$ OR $q_{(n+m+1)}$
OPERATING SYSTEM (a) OUTPUT QUANTITY
(n+m+1) QUANTITY (n+m+1)
d
$q_{(out)(osa)}$ OR $q_{(osa)}$ OR $q_{(d)}$ OR $q_{(n+m+1)}$

Notes:

1. [PO] is the symbol for the generalized performance operator that describes the effect of the operating component, operating sub-system or operating system in establishing functional relationships between input and output quantities. (The operator may have many forms, including explicit mathematical expressions, empirical curves, experimental data, or any combination of these.)
2. Letter subscripts on performance operator symbols identify the operating component, sub-system or system for which the operator describes performance. Combination subscripts are used with (out) to show that an output quantity is involved and a number with no hyphen to associate the operator with the operating component identified by the number.
3. Number subscripts without any associated letter subscripts on performance operator symbols identify the chain connection points between which the operator describes performance. The chain connection point at which the independent quantity for the operator occurs has its identifying number in the subscript position closest to the operator symbol. The chain connection point number associated with the dependent quantity for the operator is placed in the subscript to the right of the independent quantity chain connection point number and is separated from it by a hyphen.
4. A single performance operator for an operating component between chain connection points i and (i + 1) means that $q_{(i+1)} = [PO]_{[i-(i+1)]}q_i$ where in general the relationship may have vectór properties (shown by placing a bar above the [PO] to give $\overline{[PO]}$ as the vector performance operator), or it may describe only a scalar relationship which may range from experimental data to well behaved but complex mathematical expressions.
5. A combination of operating components acting between typical points i and (i + 2) of an operating chain is written as

$$[PO]_{[i-(i+2)]} = [PO]_{[(i+1)-(i+2)]}[PO]_{[i-(i+1)]}.$$

This expression implies in general that the operators are applied to the input quantities in the proper order to give the correct over-all input-output relationships for the operating component combination.
In formal terms,

$$q_{(i+1)} = [PO]_{[i-(i+1]}q_{(i)} \qquad q_{(i+2)} = [PO]_{[(i+1)-(i+2)]}q_{(i+1)}$$

so that

$$q_{(i+2)} = [PO]_{[(i+1)-(i+2)]}[PO]_{[i-(i+1)]}q_{(i)}$$

or

$$[PO]_{[i-(i+2)]} \equiv [PO]_{[(i+1)-(i+2)]}[PO]_{[i-(i+1)]}$$

If vector relationships are involved, the performance operators can *not* in general be combined without regard to the order of application, i.e.,

$$q_{(i+2)} = \overline{[PO]}_{[(i+1)-(i+2)]}\overline{[PO]}_{[i-(i+1)]}q_{(i)} \neq \overline{[PO]}_{[i-(i+1)]}\overline{[PO]}_{[(i+1)-(i+2)]}q_{(i)}$$

If only scalar relationships are involved, and all the operating components have linear mathematical characteristics,* the performance operators may usually be combined without restrictions on the order of application.
Performance operators which may be combined without regard to order and may be represented in terms of linear forms are ***performance functions***. Performance functions are given the main symbol [PF] instead of the [PO] which represents the generalized performance operator.

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* See Chapter I "Advanced Calculus for Engineers" Hildebrand (21).

Fig. 3-3. Performance operators and performance functions for combinations of operating components making up an open-chain operating system with operating subsystems.

The comparator operates to take the algebraic difference between the input signal and the feedback signal and to produce the feedback correction signal, which represents the change in the feedback signal required to make it identical with the input signal. The feedback correction signal is represented by the symbol(Sg) (for signal), placed after the symbol C (for correction) enclosed in parentheses and carrying the subscript (fb) to show that the correction is for the feedback signal. The feedback correction signal is the input for the output signal correction generating system. The output from this component is the output signal correction signal, represented by a signal symbol followed by a bracket containing the correction symbol (C) (indicating a correction for the quantity whose symbol follows the closing parenthesis) before the symbol for the output signal. The output signal generating system receives the output signal correction signal and produces two outputs. The direct chain output is the output signal that goes to the output generating system, which produces the output quantity from the over-all system. The feedback chain output from the output signal generating system is the output feedback signal, which acts as the actuating input for the feedback signal generating system. The output from this system is the feedback signal that has already been noted as one of the two inputs for the input signal – feedback signal comparator. In Fig. 3-4, the feedback signal generating system is shown as receiving an indefinite number of modifying inputs that affect the feedback signal.

3.8. The over-all action of the system of Fig. 3-4 is to produce an output that varies as some function of the actuating input quantity and the modifying inputs. The closed-chain organization for a system has a number of operating advantages over an open-chain system designed to accomplish like results. In particular, a closed-chain system affords a much closer control of the relationship between an output quantity and a group of input quantities than is generally possible with operating components of given performance organized in an open-chain arrangement.

3.9. Equations (1) to (7) in Fig. 3-4 give expressions for the outputs from individual operating components in terms of their inputs. Equations (9) to (14) of this figure are the expressions for various quantities associated with operation of the system in terms of the actuating input. Equations (15) to (22) are the performance functions that describe the action of certain combinations of operating components in establishing relationships between quantities at specified chain connection points of the system. For example, the feedback correction signal – output quantity performance function describes the action of the system in establishing the relationship between the feedback correction signal and the output when the feedback chain is complete as shown in Eq. (17), Fig. 3-4. This performance function is specified by two subscripts to the main symbol (PF). The first subscript is written as (c-o), in which the first symbol is identified with the chain

ACTUATING INPUT QUANTITY
FIRST QUANTITY ①
$q_{(in)(ac)}$
$q_{(in)}$
q_1
i
INPUT RECEIVER (ir) OR (1)
$[PF]_{(ir)}$ OR $[PF]_{(1-2)}$ OR $[PF]_1$
INPUT SIGNAL
SECOND QUANTITY ②
$(Sg)q_{(in)}$
q_2
INPUT SIGNAL-FEEDBACK SIGNAL COMPARATOR (isfsc) OR (2)
$[PF]_{(isfsc)(dir)}$ * OR $[PF]_{(2-3)}$ OR $[PF]_{(2)(dir)}$ OR $[PF]_{(isfcs)(fb)}$ ** OR $[PF]_{(8-3)}$ OR $[PF]_{(2)(fb)}$
FEEDBACK CORRECTION SIGNAL
THIRD QUANTITY ③
$(C)(Sg)_{(fb)}$
c
q_3
OUTPUT SIGNAL CORRECTION GENERATING SYSTEM (oscgs) OR (3)
$[PF]_{(oscgs)}$ OR $[PF]_{(3-4)}$ OR $[PF]_{(3)}$
OUTPUT SIGNAL CORRECTION SIGNAL
FOURTH QUANTITY ④
$(Sg)\{(C)[(Sg)q_{(out)}]\}$
q_4
OUTPUT SIGNAL GENERATING SYSTEM (osgs) OR (4)
$[PF]_{(osgs)(dir)}$ * OR $[PF]_{(4-5)}$ OR $[PF]_{(4)(dir)}$ OR $[PF]_{(osgs)(fb)}$ ** OR $[PF]_{(4-7)}$ OR $[PF]_{(4)(fb)}$
OUTPUT SIGNAL
FIFTH QUANTITY (5)
$(Sg)q_{(out)}$
q_5
OUTPUT GENERATING SYSTEM (ogs) OR (5)
$[PF]_{(ogs)}$ OR $[PF]_{(5-6)}$ OR $[PF]_{(5)}$
OUTPUT QUANTITY
(6) SIXTH QUANTITY
o
$q_{(out)}$
q_6
OUTPUT FEEDBACK SIGNAL
⑦
$[(Sg)q_{(out)}]_{(fb)}$
⑦ SEVENTH QUANTITY
q_7
FEEDBACK SIGNAL GENERATING SYSTEM (fsgs) OR (7)
$[PF]_{(fsgs)}$ OR $[PF]_{(7-8)}$ OR $[PF]_{(7)}$
⑧ EIGHTH QUANTITY
q_8
⑧ fb
FEEDBACK SIGNAL
$(Sg)_{(fb)}$
FIRST FEEDBACK MODIFYING INPUT
$q_{(mod)(fb)1}$
nth FEEDBACK MODIFYING INPUT
$q_{(mod)(fb)n}$

* The subscript (dir) shows that the performance function to which it is attached represents the direct chain operation of the operating component involved.

** The subscript (fb) shows that the performance function to which it is attached represents the feedback chain operation of the operating component involved.

Notes:

1. The symbol **(Sg)** before a quantity symbol denotes a signal associated with the quantity represented by the symbol. The symbol **(Sg)** with a subscript, but without a following quantity symbol, denotes a signal associated with the chain point or operating component identified by the subscript symbol.
2. This diagram represents the situation in which the performance function of the feedback signal generating system depends upon an indefinite number, *n*, of feedback modifying inputs.
3. The performance function of each operating component is written with three types of identifying subscripts:
   - a) An all letter subscript associating the performance function with the letter identification of an operating component.
   - b) A subscript made up of two numbers, separated by a hyphen and enclosed in parentheses, which associate the performance function with two chain connection points. The order in which these numbers are written corresponds to the direction of progression from the input chain connection point.
   - c) A single number subscript associating the performance function with the identifying number of an operating component.
4. In cases where several performance functions are associated with an operating component, additional defining subscripts are used. For example, the subscripts (dir) and (fb) show that the performance functions to which they are attached represent direct chain operation and feedback chain operation, respectively, of the operating component involved.
5. This group of equations is written under the assumption that the performance functions involved are expressed by forms that may be manipulated by the rules of conventional algebra. (Refer to par. 3.5 in Chapter 3.)

(1) $(Sg)q_{in} = [PF]_{ir}\,q_{in}$

(2) $(C)(Sg)_{fb} = [PF]_{(isfsc)dir}[(Sg)q_{in}] - [PF]_{(isfsc)fb}[(Sg)_{fb}]$

(3) $(Sg)\{(C)[(Sg)q_{out}]\} = [PF]_{oscqs}[(C)(Sg)_{fb}]$

(4) $[(Sg)q_{out}]_{fb} = [PF]_{(osqs)fb}(Sg)\{(C)[(Sg)q_{out}]\}$

(5) $(Sg)_{fb} = [PF]_{fsqs}[(Sg)q_{out}]_{fb}$

(6) $(Sg)q_{out} = [PF]_{(osqs)dir}(Sg)\{(C)[(Sg)q_{out}]\}$

(7) $q_{out} = [PF]_{oqs}[(Sg)q_{out}]$

Expressing all quantities in terms of the input quantity q_{in}

(8) $(C)(Sg)_{fb} = [PF]_{(isfsc)dir}[PF]_{ir}q_{in} - [PF]_{(isfsc)fb}[PF]_{fsqs}[PF]_{(osqs)fb}[PF]_{oscqs}[(C)(Sg)_{fb}]$

or

(9) $$(C)(Sg)_{fb} = \frac{[PF]_{ir}[PF]_{(isfsc)dir}}{1 + [PF]_{(isfsc)fb}[PF]_{oscqs}[PF]_{(osqs)fb}[PF]_{fsqs}}\,q_{in}$$

(10) $$(Sg)\{(C)[(Sg)q_{out}]\} = \frac{[PF]_{ir}[PF]_{(isfsc)dir}[PF]_{oscqs}}{1 + [FF]_{(isfsc)fb}[PF]_{oscqs}[PF]_{(osqs)fb}[PF]_{fsqs}}\,q_{in}$$

(11) $$[(Sg)q_{out}]_{fb} = \frac{[PF]_{ir}[PF]_{(isfsc)dir}[PF]_{oscqs}[PF]_{(osqs)fb}}{1 + [PF]_{(isfsc)fb}[PF]_{oscqs}[PF]_{(osqs)fb}[PF]_{fsqs}}\,q_{in}$$

(12) $$(Sg)_{fb} = \frac{[PF]_{ir}[PF]_{(isfsc)dir}[PF]_{oscqs}[PF]_{(osqs)fb}[PF]_{fsqs}}{1 + [PF]_{(isfsc)fb}[PF]_{oscqs}[PF]_{(osqs)fb}[PF]_{fsqs}}\,q_{in}$$

(13) $$(Sg)q_{out} = \frac{[PF]_{ir}[PF]_{(isfsc)dir}[PF]_{oscqs}[PF]_{(osqs)dir}}{1 + [PF]_{(isfsc)fb}[PF]_{oscqs}[PF]_{(osqs)fb}[PF]_{fsqs}}\,q_{in}$$

(14) $$q_{out} = [PF]_{ir}[PF]_{oqs}\left[\frac{[PF]_{(isfsc)dir}[PF]_{oscqs}[PF]_{(osqs)dir}}{1 + [PF]_{(isfsc)fb}[PF]_{oscqs}[PF]_{(osqs)fb}[PF]_{fsqs}}\right]q_{in}$$

or

(15) $$[PF]_{(i\text{-}o)cc} \equiv \left(\frac{q_{out}}{q_{in}}\right)_{cc} = [PF]_{ir}[PF]_{oqs}\left\{\frac{[PF]_{(isfsc)dir}[PF]_{oscqs}[PF]_{(osqs)dir}}{1 + [PF]_{(isfsc)fb}[PF]_{oscqs}[PF]_{(osqs)fb}[PF]_{fsqs}}\right\}$$ ≡ Input – output closed-chain performance function

(16) $[PF]_{(i\text{-}c)oc} \equiv [PF]_{ir}[PF]_{(isfsc)dir}$ ≡ Input quantity – feedback correction signal open-chain performance function

(17) $[PF]_{(c\text{-}o)} \equiv [PF]_{oscqs}[PF]_{(osqs)dir}[PF]_{oqs}$ Feedback correction signal – output quantity performance function (for either open- or closed-chain operation)

(18) $[PF]_{[fb\text{-}o]oc} \equiv [PF]_{(isfsc)fb}[PF]_{oscqs}[PF]_{(osqs)dir}[PF]_{oqs} = [PF]_{(isfsc)fb}[PF]_{c\text{-}o}$ ≡ Feedback signal – output quantity open-chain performance function

(19) $[PF]_{(i\text{-}o)oc} \equiv [PF]_{ir}[PF]_{(isfsc)dir}[PF]_{oscqs}[PF]_{(osqs)dir}[PF]_{oqs} = [PF]_{ir}[PF]_{[fb\text{-}o]oc} = [PF]_{(i\text{-}c)oc}[PF]_{c\text{-}o}$

≡ Input quantity – output quantity open-chain performance function

(20) $$[PF]_{[fb\text{-}o]cc} \equiv \frac{[PF]_{(osqs)dir}[PF]_{oqs}}{[PF]_{(osqs)fb}[PF]_{fsqs}}$$ ≡ Feedback signal – output quantity closed-chain performance function

(21) $$[PF]_{(i\text{-}c)cc} \equiv \frac{[PF]_{ir}[PF]_{(isfsc)dir}}{1 + [PF]_{(isfsc)fb}[PF]_{oscqs}[PF]_{(osqs)fb}[PF]_{fsqs}} = \frac{[PF]_{(i\text{-}c)oc}}{\left[1 + \dfrac{[PF]_{[fb\text{-}o]oc}}{[PF]_{[fb\text{-}o]cc}}\right]}$$ ≡ Input quantity – feedback correction signal closed-chain performance function

(22) $$[PF]_{(i\text{-}o)cc} \equiv [PF]_{ir}[PF]_{oqs}\left\{\frac{[PF]_{(isfsc)dir}[PF]_{oscqs}[PF]_{(osqs)dir}}{1 + [PF]_{(isfsc)fb}[PF]_{oscqs}[PF]_{(osqs)fb}[PF]_{fsqs}}\right\} = \frac{[PF]_{(i\text{-}o)oc}}{\left[1 + \dfrac{[PF]_{[fb\text{-}o]oc}}{[PF]_{[fb\text{-}o]cc}}\right]}$$

$$= [PF]_{(i\text{-}o)oc}\frac{[PF]_{(i\text{-}c)cc}}{[PF]_{(i\text{-}c)oc}}$$ ≡ Input quantity – output quantity closed-chain performance function

Fig. 3-4. Performance characteristics of a generalized system in terms of performance functions for open-chain and closed-chain operation.

connection point shown by the small letter c in a square box. This letter is chosen to suggest the word "correction," which is here associated with the feedback correction. The symbol that follows the hyphen after the c of the first subscript parentheses is o for the square box o of Fig. 3-4, located at the system output. Following the chain connection point identifying subscript parentheses, a second subscript consisting of the letters cc can be used to show that the identified performance operator describes the behavior of the system when the feedback chain is closed, as shown in Fig. 3-4.

FEEDBACK CORRECTION - OUTPUT CLOSED-CHAIN PERFORMANCE FUNCTIONS

3.10. For the purposes of analysis, design and experimental test, it is useful to define an open-chain performance function (Eq. (18) of Fig. 3-4) corresponding to the feedback signal – output quantity closed-chain performance function (Eq. (20) of Fig. 3-4). This open-chain performance function describes the action of the system of Fig. 3-4 when the functional chain is broken at the output of the feedback signal generating system. In general, the open-chain performance function describes the action of the system when the functional chain is broken at the output of the component which produces the quantity identified by the symbol in the chain connection point subscript nearest the main symbol (see Eq. (18) of Fig. 3-4). The symbol for the performance function representing the action of this combination of components is identical with that for the corresponding closed-chain situation except that the second subscript is (oc), to suggest the words "open chain," instead of the subscript (cc), which is intended to suggest "closed chain." Equations (15) to (22) of Fig. 3-4 give a number of the open- and closed-chain performance functions that act between various chain connection points of this figure. For reference purposes, relationships among the various performance functions are also given in this group of equations. In practice, performance functions may be expressed in many ways. The particular form to be selected for any given situation must depend upon a considerable background of knowledge and experience. A discussion of the possible ways of writing performance functions is not required for present purposes, which are concerned only with the general procedure used to form performance functions for generalized combinations of operating components. The various methods for representing performance functions by mathematical expressions are given in detail in later chapters.

CHAPTER 4

STATIC AND DYNAMIC OPERATING CONDITIONS; STATIC SENSITIVITY

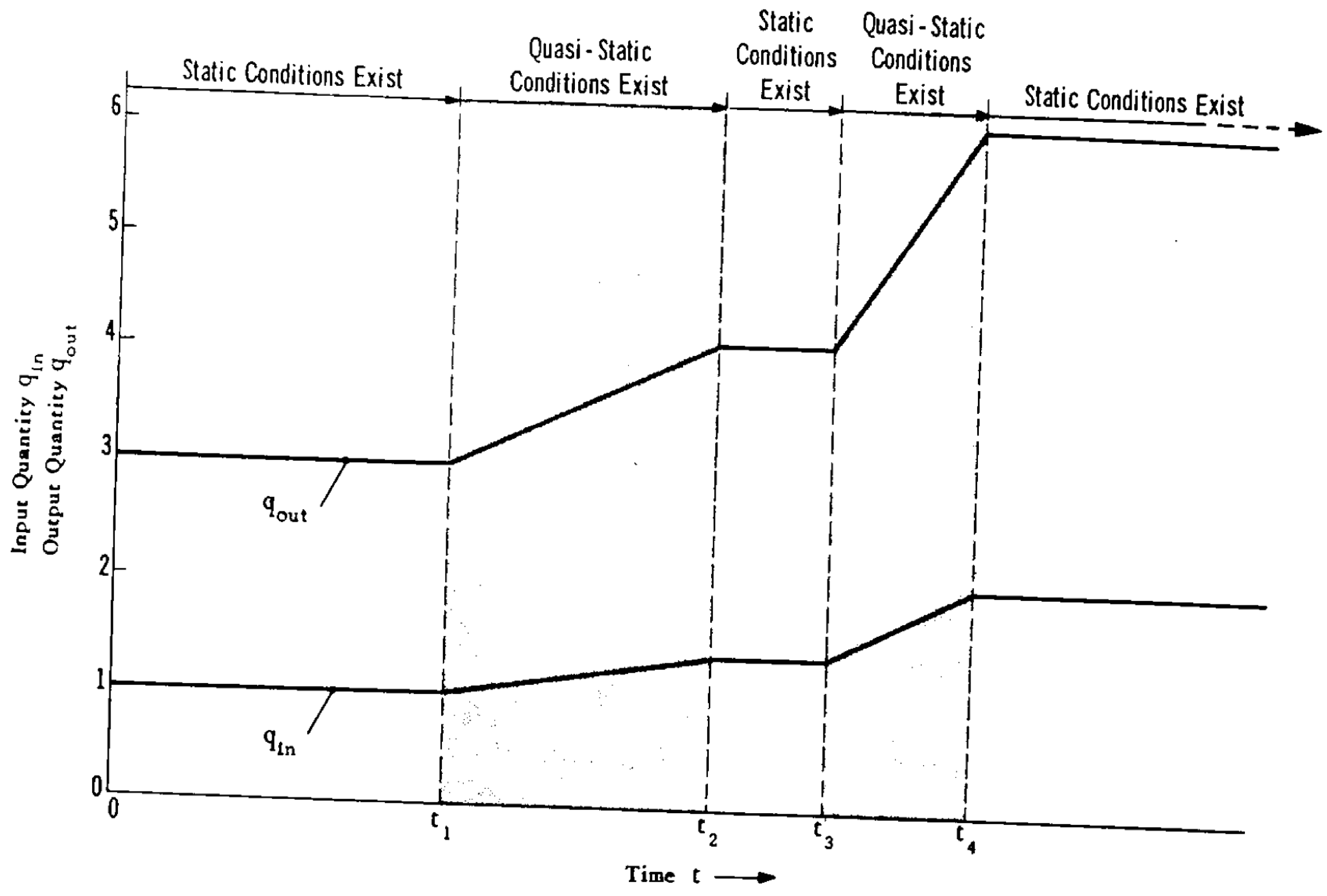
STATIC CONDITIONS AND STATIC SENSITIVITY

4.1. Operating components establish relationships between input and output quantities that, in general, depend upon the way in which the quantities involved are changing with time. This means that performance operators and performance functions are not fixed for a given operating component, but are influenced by the rapidity of input and output variations.

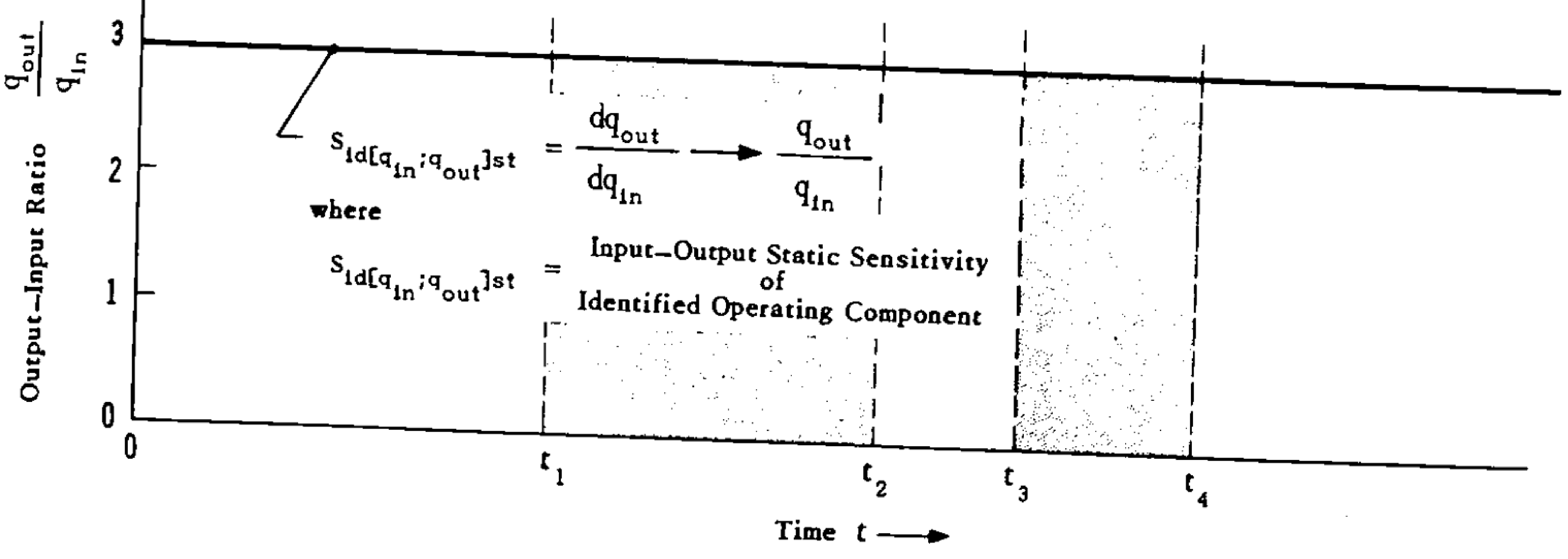
4.2. Static conditions exist for operating components when the input and output quantities are both constant. In this situation, the scalar performance operator and the performance function both reduce to the static sensitivity, which by definition is the ratio of a small change between constant output levels to the corresponding small change between constant input levels. This definition is illustrated by the plots of Fig. 4-1 for the special case of an operating component having constant static sensitivity. Under this assumption, the expression for static sensitivity reduces to the simple ratio of the output to the input. The section of the plots in Figs. 4-1a and 4-1b between $t = 0$ and $t = t_1$ illustrates typical static conditions with both the input and output levels constant. The plot of Fig. 4-1b gives the static sensitivity as the ratio of the output to the input. In the plot of Fig. 4-1a, the output and input quantities vary with time between $t = t_1$ and $t = t_2$. The plot of Fig. 4-1b shows that the static sensitivity was unchanged by the input and output variations. Whenever conditions of this type exist, in which the input-output relationship is unchanged from the static sensitivity when the quantities involved are changing with time, the operating component is working under quasi-static conditions. In the plots of Fig. 4-1, the input-output ratio, which is equivalent to the static sensitivity, is shown as constant not only during the static condition intervals between 0 and t_1, t_2 and t_3, and after t_4, but also during the intervals t_1 to t_2, and t_3 to t_4. This means that the variations of Fig. 4-1a all fall within static and quasi-static conditions for the operating components considered.

COMPLETE SYMBOLS FOR PERFORMANCE-DESCRIBING CONCEPTS

4.3. Static sensitivity is shown in Fig. 4-1 as a completely specified performance-describing concept. Previous definitions have carried identification for the operating component concerned, but did not include any specification of



a) Input and output variations under static and quasi-static conditions



b) Input – output static sensitivity under static and quasi-static conditions

Note:

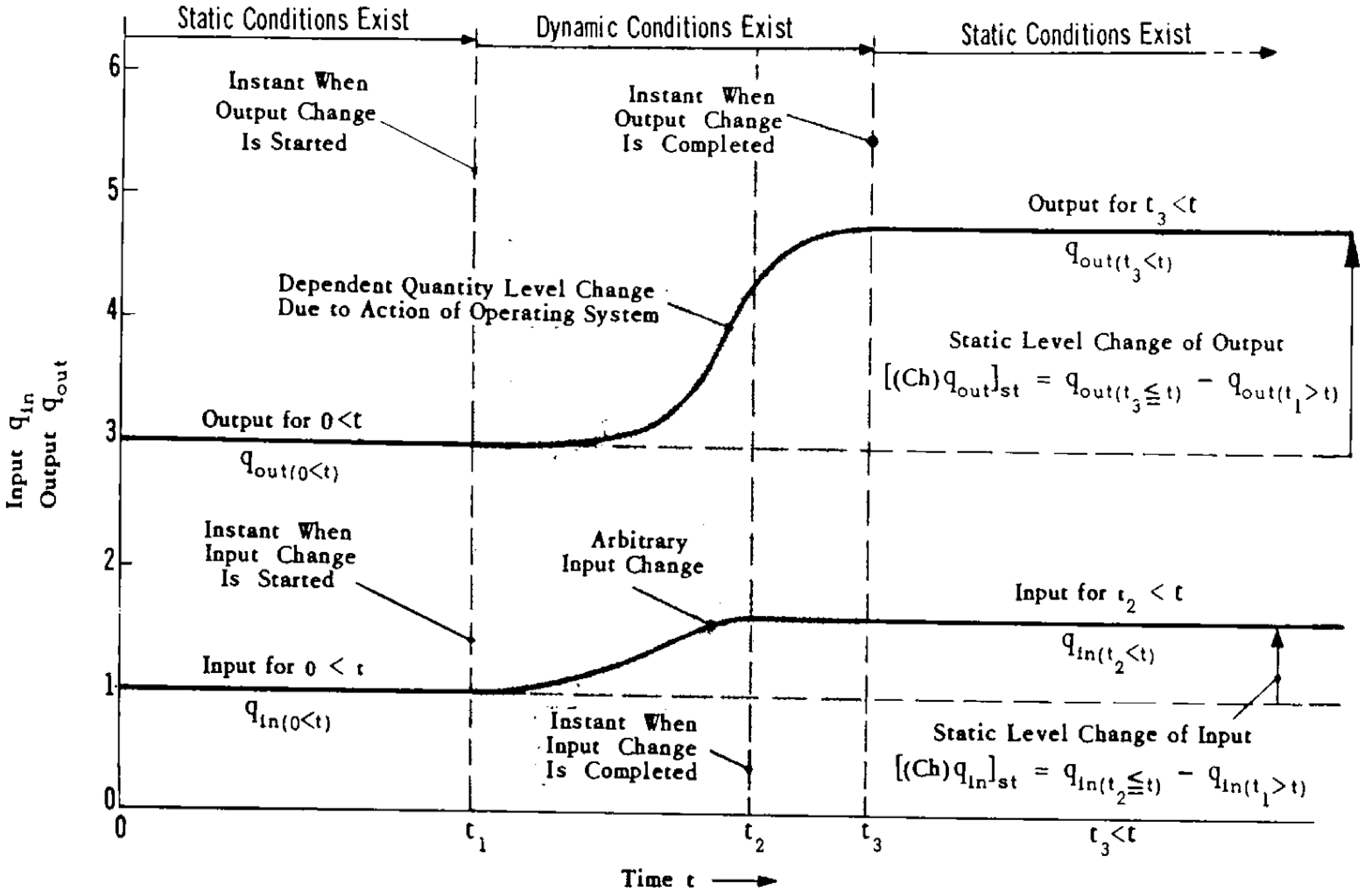
1. By definition, ***static sensitivity*** is the ratio of small changes between constant levels of the ***output quantity*** to the corresponding small changes between constant levels of the ***input quantity.***
2. This relationship is a special case in which the output is a linear function of the input.
3. 1) The ***main symbol*** S represents ***Sensitivity.***
   2) The first subscript parenthesis symbol identifies the ***operating component*** whose static performance is described by the complete symbol.
   3) The square subscript bracket is the ***input-output bracket.***
   4) The subscript symbol inside the square bracket nearest the main symbol represents the ***input quantity.***
   5) The subscript symbol inside the square bracket separated from the ***input quantity*** symbol by a semi-colon represents the ***output quantity***
   6) The subscript (st) following the input-output bracket identifies the sensitivity as being associated with ***static conditions.***
4. The present application of the term ***Sensitivity*** is a special case of a much broader definition given later in the book.

Fig. 4-1. Illustrative input and output variations for an operating component under static and quasi-static conditions. Definition of static sensitivity.

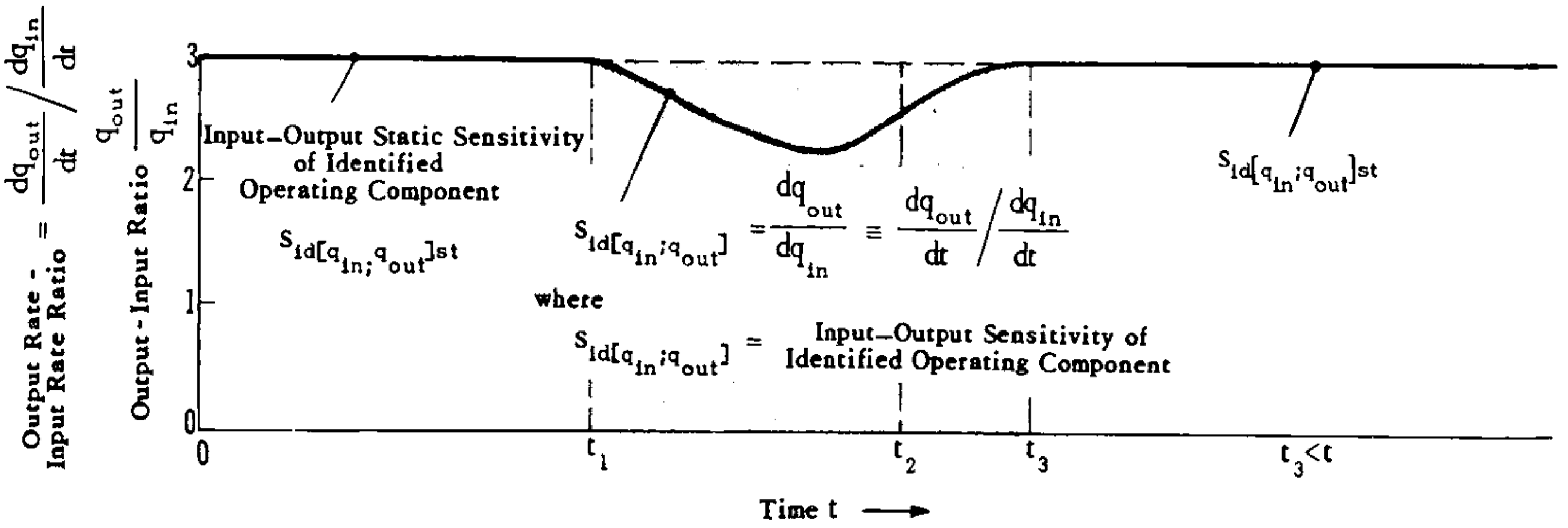
the input and output quantities considered. This specification is provided by adding the input-output bracket to the other subscripts of the symbol for a performance-describing concept. The use of the input-output bracket and other subscripts for static sensitivity is illustrated and explained in Fig. 4-1.

DYNAMIC CONDITIONS

4.4. Figure 4-2 illustrates the application of the sensitivity concept to situations in which the ratio of the output to the input differs from the static sensitivity during a period of input and output changes. The concept may be extended to



a) Input and output variations under static and dynamic conditions



b) Input – output sensitivity under static and dynamic conditions

Fig. 4-2. Illustrative input and output variations for an operating component under static and dynamic conditions.

include periods of this kind by defining *sensitivity** (without the qualifying word "static") as the ratio of the time rate of change of the output to the time rate of change of the input. The symbol for this generalized sensitivity is identical with that for static sensitivity except for the omission of the subscript (st). The plot of Fig. 4-2 illustrates a situation in which the sensitivity differs from the static sensitivity during the interval between t_1 and t_3, when input and output variations are taking place. When a situation of this kind exists, the operating component concerned is said to be operating under *dynamic conditions.* An equivalent statement is that dynamic conditions exist when the input-output relationship established by a given operating component depends on the variation rates in the output and input quantities.

SENSITIVITY, SCALE FACTOR AND GRADIENT

4.5. It is of interest to note here that sensitivity as it is defined above is similar to the concept commonly called *scale factor*, or *gradient*. No difficulty is introduced by this duplication of names if the concepts involved are completely understood. However, it appears that the term *sensitivity* is more naturally associated with the operation of an instrument or control system than either the term *scale factor*, which suggests the number by which the actual scale reading of a measuring instrument must be multiplied to find the indicated value of the actuating input, or *gradient*, which is generally accepted as describing the steepest slope** associated with contour lines on a multidimensional model. In symbol form, scale factor is indicated by the subscript f (for scale factor) to the main sensitivity symbol S. Additional identifying subscripts are then added as required, and as already described.

4.6. It is a long-established practice in instrument engineering to associate the term "sensitivity" with the operation of measuring instruments, with the quantities to be measured as their inputs and index positions as their outputs. For example, the *pressure input – index angle output sensitivity* of the pressure gage illustrated in Fig. 2-2 might be sixty degrees per pound per square inch.

SENSITIVITY AND GAIN

4.7. Common practice in describing the static performance of electrical systems, particularly if electronic equipment is involved, is to use the concept of *gain*. In general, this concept is identical with that of sensitivity. Gain is often dimensionless in the sense that the input and output quantities considered have identical dimensions. For example, the *voltage input – voltage output gain* of a given amplifier might be one hundred. Gain may also have dimensions in the sense that an amplifier might operate with a *voltage input – current output gain* of one milliampere per volt. In this book the capital letter G is used as the main symbol for *gain*, and is in all ways equivalent to the concept of sensitivity with the main symbol S. The system of identifying and specifying subscripts developed for association with the sensitivity symbol S is applied without modification to the symbol G when this letter serves as the main symbol for gain. This same symbol, G, may be used to represent *gradient* when it is desired to use this term for the sensitivity concept. Similarly, the main symbol (SF) suggests scale factor when this term is used.

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* This definition of sensitivity is provisional only. A complete definition of sensitivity is given in Chapter 10.

** Gradient is the name applied to a specific concept in vector analysis. See, for example, section 116 in Franklin (17).

CHAPTER 5

STATIC PERFORMANCE

DESCRIPTION OF STATIC PERFORMANCE

5.1. For the purpose of instrument engineering it is desirable to establish concepts, procedures and conventions for describing the static performance of operating components and systems. Any satisfactory method for working out a description of this kind should produce data in a form adapted to transfer qualitative and quantitative static performance information by direct inspection and very simple calculations.

5.2. It is an almost universal characteristic of instrument systems that their regions of satisfactory operation are bounded by regions of unsatisfactory operation. In any given case the regions of satisfactory and unsatisfactory operation may be specified in terms of the operating level ratio, as a dimensionless variable. By definition this quantity is the ratio of the level of a properly chosen input or output quantity to some convenient reference level of this same quantity. For example, the operating level ratio for the pressure gage of Fig. 2-2 might be defined in terms of the input pressure to the gage, with a pressure level of one, ten, or some other convenient number of pressure units within the region of satisfactory operation taken as the reference. For some regions of large pressure level ratio values the gage must cease to operate because of interferences among parts, arbitrarily imposed stops, or actual failures of mechanical components.

HIGH INPUT LIMIT OF OPERATION

5.3. When the pressure level ratio increases, but before gage operation ceases entirely, the pressure input – index angle output ratio (i.e., the static sensitivity) will generally deviate from the value considered as satisfactory. This effect will in general be caused by overloading, or saturation, of the gage mechanism. If the pressure level is increased beyond the overloading region until the index no longer responds, the gage enters the high input nonoperating region.

LOW INPUT LIMIT OF OPERATION

5.4. Decrease of the pressure level ratio for the pressure gage from the overloading region to the operating region allows the gage to perform properly until the pressure level ratio reaches the interference level region. Within the interference level region effects due to the pressure input become so small that they are comparable to interference effects caused by bearing friction, binding of mechanical parts, etc. This means that the index position is no longer controlled by the pressure input alone, but also by effects inside the mechanism, so that the static sensitivity will deviate from its value in the operating region. As the pressure level ratio decreases from the interference zone limit, the index will no longer respond to changes of the input pressure. When this occurs, the gage is inoperative. Pressure level changes too small to cause index motions define the low input nonoperating region.

PERFORMANCE RATIO

5.5. The concept of operating level ratio provides a means for identifying the regions associated with the operation of any component or system. It remains to define a concept for describing behavior under static conditions as the operating level ratio changes from the interference region through the operating region to the overloading region. To be useful in all situations this concept should be based on a direct comparison of the actual static performance for any given operating level ratio with that static performance considered as satisfactory without regard for the quantities involved or the levels at which operation is examined. These characteristics are provided by the performance ratio, which is defined for a given operating level as the ratio of a number representing the performance actually existing to a number that represents an arbitrarily selected reference performance. For example, if static sensitivity is chosen as the performance-describing quantity, the static sensitivity performance ratio for any given operating level is the ratio of the actual static sensitivity to a selected reference static sensitivity. In practice this reference static sensitivity would be chosen as the ideal operating region static sensitivity for the component or system under consideration. In many cases this ideal static sensitivity will be constant, while in other cases it will vary in some way with the operating level ratio. The concept of performance ratio applies to all situations, no matter how the reference sensitivity is chosen. In general there are no hard-and-fast rules that can be used in choosing this reference beyond the principle that it should allow an effective description in simple numbers of the performance characteristics for the system under consideration.

5.6. The main symbol for performance ratio is (PR) followed by a first subscript identifying the operating component or system with which the performance ratio is associated, and a second subscript showing the performance-describing concept used in forming the performance ratio. These conventions are illustrated in Fig. 5-1.

5.7. Performance ratio curves based on properly chosen reference performance measures are often so close to unity within the operating region that it is impossible to make sufficiently accurate estimates of performance ratio range from zero to unity. It is generally desirable to show performance ratio plots on scales of this type in order to convey "at a glance" information on percentage variations in performance. This means that the zero-to-unity performance ratio scale should be included in performance plots. However, with a scale of this kind, small-magnitude variations of the performance ratio cannot be accurately read from the curve, so that auxiliary plots must be provided to show this information. Small magnitude variations may be satisfactorily exhibited by plots of the performance ratio deviation, which is by definition equal to the performance ratio

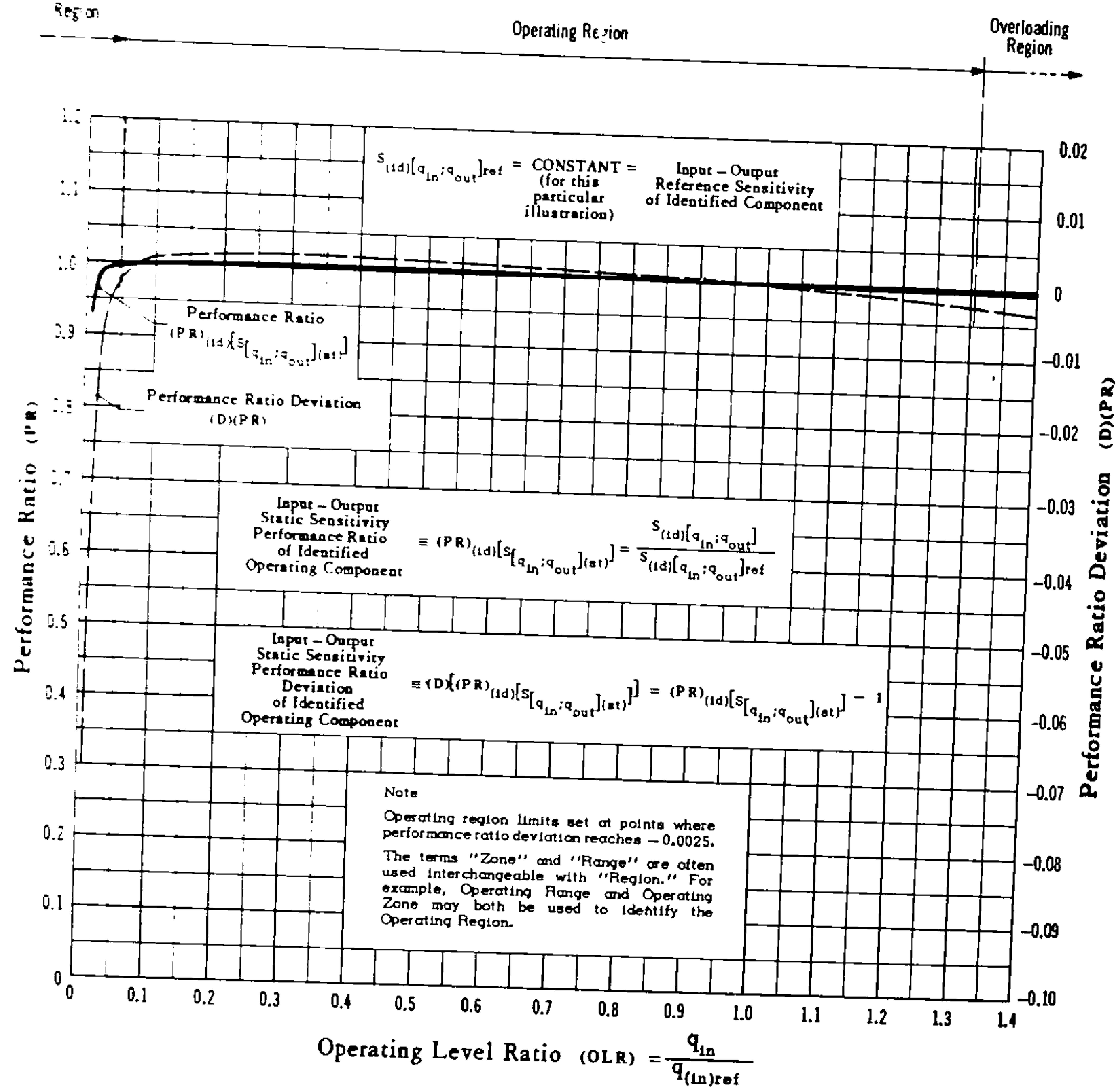

Fig. 5-1. Typical linear scale plots of static sensitivity performance ratio and performance ratio deviation.

minus unity. Because in practice deviations of this kind are usually much smaller than the performance ratio, it is possible to place a greatly expanded deviation scale in the space used for the zero-to-unity range for the performance ratio. For convenience, the zero deviation axis is often made coincident with the unity axis of the performance ratio plot, and the deviation scale is chosen to have one, one-tenth, one-hundredth, etc., of one unit in the distance assigned to one unit of the performance ratio scale. When these conventions are used, deviation plots in effect "put a microscope" on variations in the performance ratio, while leaving the operating level scale unchanged. The symbol for performance ratio deviations is a capital D (for deviation), in parentheses, placed before the complete symbol for performance ratio.

5.8. Figure 5-1 shows typical static performance-describing plots with linear scales for the operating level ratio abscissa and the performance ratio and performance ratio deviation ordinates. In this figure the symbols are formed in accordance with the conventions outlined above, with the component identification and the input-output bracket subscripts containing generalized symbols. Complete names and defining equations for various symbols are given in the figure. The reference sensitivity has been selected as a constant for the curve of Fig. 5-1. The value of this constant was chosen to substantially match the level of the static sensitivity within the operating region for the operating component considered. The full-line curve shows that the performance ratio is effectively unity in the operating region. The operating level ratio for the lower boundary of the operating region is determined by the point at which the performance ratio shows a marked variation from its operating region value. This boundary is marked by a dashed line in Fig. 5-1, and is shown as the upper boundary for the interference region. No departure of the performance ratio curve from unity to determine the lower boundary for the saturation (overloading) region is detectable in Fig. 5-1. On the other hand the performance ratio deviation curve shown by the thin dashed line in Fig. 5-1, which is plotted with one-hundredth of a unit given the same space as one-tenth of a unit of the performance ratio curve, does show the beginning of a departure from operating region performance at an operating level ratio of about 1.3. The convention has been adopted here of using a full-line curve when referring to coordinates on the left side of the plot, and a dashed-line curve when referring to coordinates on the right side of the plot.

TOLERANCE LIMITS

5.9. Any definition of the operating region must be based on tolerance limits arbitrarily set by the performance requirements for the operating component being described. For the illustrative plots of Fig. 5-1, the dashed line showing the lower and upper limits for the operating zone are set at the operating level ratios for which the performance ratio deviation has reached a value of minus 0.0025. It is obvious that the operating region will vary as the performance ratio deviation tolerance limits are changed.

LOGARITHMIC SCALE PERFORMANCE RATIO PLOTS

5.10. The plots of Fig. 5-1 are not generally satisfactory for describing performance in the high and low operating level regions, because of the very small abscissa distance available between zero and the interference region limit and the difficulty of showing high operating level ratio values when a great range of this level intervenes before the overloading region is reached. For these reasons linear scale plots do not lend themselves well to determinations of operating

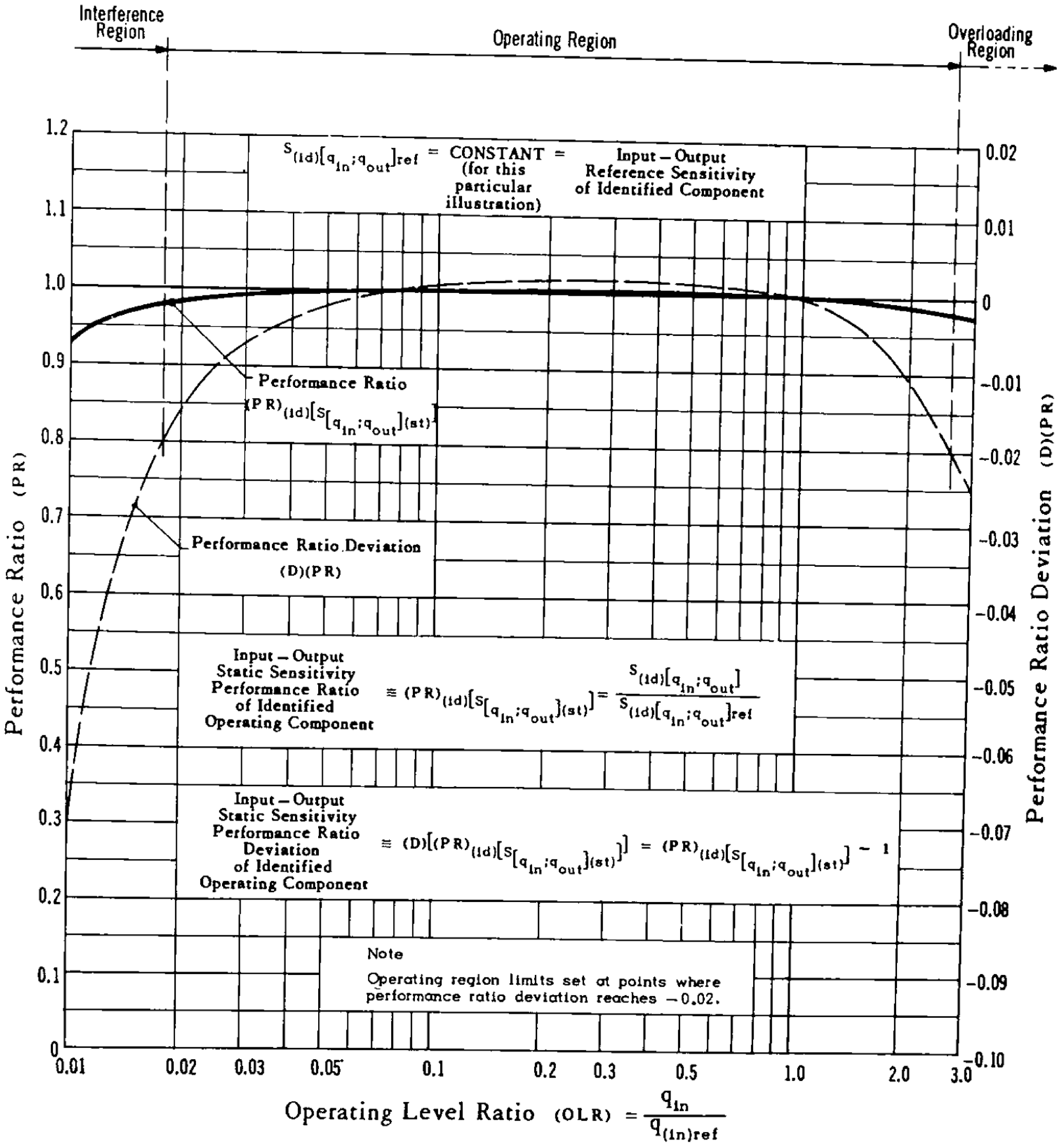

Fig. 5-2. Typical semi-logarithmic plots of static sensitivity performance ratio and performance ratio deviation.

regions. Any change of the linear operating level scale that does not introduce discontinuous breaks for the purpose of showing more detail outside the operating region would require a considerable compression of the abscissa scale. In practice this compression might be helpful at high operating levels but would make the situation worse for low operating levels.

5.11. The linear scale difficulty in the interpretation of performance ratio curves may be avoided by employing a logarithmic scale* for the operating level ratio. To illustrate the results of this change of scales, the curves of Fig. 5-1 are replotted in Fig. 5-2 with the same ordinate scales but with the scale for the operating level ratio changed from linear to logarithmic. Inspection of Fig. 5-2 shows that the operating region is compressed in distance along the abscissa scale as compared with the corresponding distance in Fig. 5-1, while the interference region and the overloading region both have a corresponding expansion. In Fig. 5-2 the performance ratio deviation is plainly shown for both high and low operating level ratios. To emphasize this point, the tolerance limits for the operating region are assumed to be set at minus 2 percent in Fig. 5-2, as compared with minus 1/4 percent in Fig. 5-1. The effect of this change in tolerance is to change the lower operating region limit from 0.048 for the 1/4-percent tolerance to 0.018 for the 2-percent tolerance. The corresponding change in the upper operating region limit is an increase from 1.3 to 2.5. This example serves to illustrate the importance of performance ratio tolerance specifications in determining the magnitude of the operating region in terms of the operating level ratio.

VARIABLE REFERENCE SENSITIVITY

5.12. Situations may arise that make it desirable to use a variable reference sensitivity for describing static performance. An example of this type is described here not only to show the features of a variable reference performance ratio plot but also to demonstrate the development of performance equations for a typical instrument system, and the use of log-log coordinates for the presentation of static performance information. The system chosen for the example is a pitot-static airspeed meter using a pressure gage similar to that described in Fig. 2-2 for the airspeed indicating head. In order to simplify the discussion, the effect of compressibility** on the pitot-static head output is neglected in the derivation of the performance equations. This omission would reduce the accuracy of results for practical application of the equations but does not interfere with a demonstration of the method. The steps of this method are described in Fig. 5-3, Performance Summary 5-1, Derivation Summary 5-1, and Fig. 5-4. Figure 5-3 describes the essential physical features of the pitot-static airspeed meter and outlines the performance characteristics for the operating components of the meter. These characteristics are given in Fig. 5-3c under the names of the components as subheadings. The letter combinations associated with the component names in the diagrams of Figs. 5-3a and 5-3b are also placed in the subheadings.

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* It is important to note that logarithms may be associated with pure numbers only. This means that all variables plotted on logarithmic coordinates must be reduced to dimensionless ratios.

** The effect of compressibility on the output of the pitot-static head is discussed in texts on aerodynamics; see, for example, Chapter 2, High-Speed Aerodynamics, by W. F. Hilton (22).

5.13. In Fig. 5-3c it is shown that the pitot-static head (symbol (psh)) produces a pressure difference. This pressure difference is the airspeed pressure (symbol p with a subscript v) which is proportional to one-half the product of the air density and the square[*] of the airspeed of the pitot-static head (symbol v with subscript (psh)). The airspeed meter mechanism receives the airspeed pressure and produces the airspeed index angle (symbol A with subscript ai). In Fig. 5-3c the performance of the meter mechanism is represented by the airspeed meter mechanism pressure — angle sensitivity (symbol S with identifying subscript (amm) and input-output bracket containing p for pressure separated from A for angle). This form does not mean that the sensitivity is constant, but it does imply that attention will be restricted to situations for which time delay effects due to the mechanism are negligible, so that static and quasi-static conditions are considered. The function of the scale in Fig. 5-3c is to relate the airspeed index position to the indicated airspeed. This action is completely described by the airspeed scale angle — airspeed sensitivity (symbol S for sensitivity, subscript (as) for airspeed scale, and with A and v in the input-output bracket).

DESIGN PERFORMANCE SUMMARY FOR AIRSPEED METER

5.14. The function of a design performance summary is to supply the essential information on the design specifications and the design performance characteristics for the complete system and its essential components. Design Performance Summary 5-1 supplies this information for the pitot-static airspeed meter of Fig. 5-3. Section (a) of Design Performance Summary 5-1 includes a symbol functional diagram for the meter and a tabulation of performance characteristics for the operating components. Section (b) of Design Performance Summary 5-1 summarizes the essential design specifications for the airspeed meter. In practice, these specifications would be given with more details than are given in Design Performance Summary 5-1b, which omits all tolerance information. Tolerances and other details are not essential for the demonstration of the performance-describing procedure under discussion, and are omitted in the interest of brevity. The basic performance specification of Design Performance Summary

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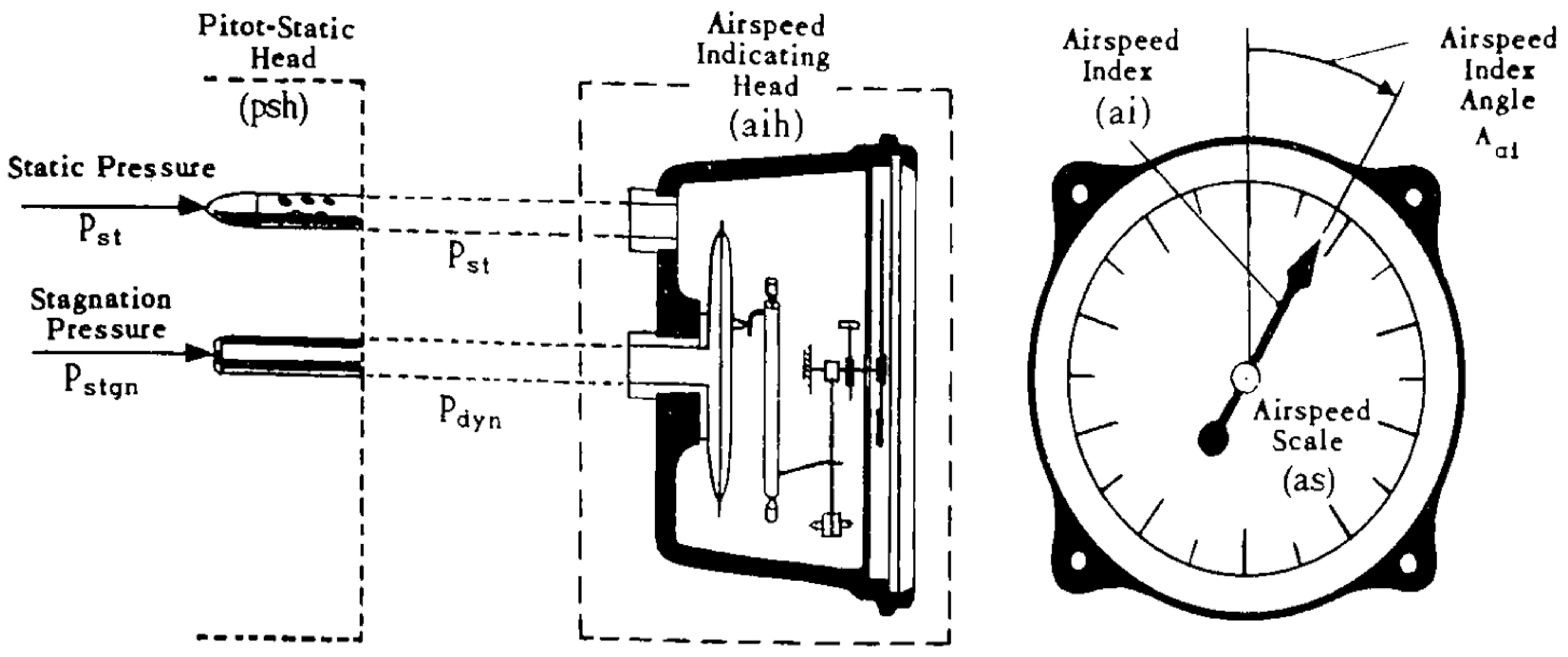
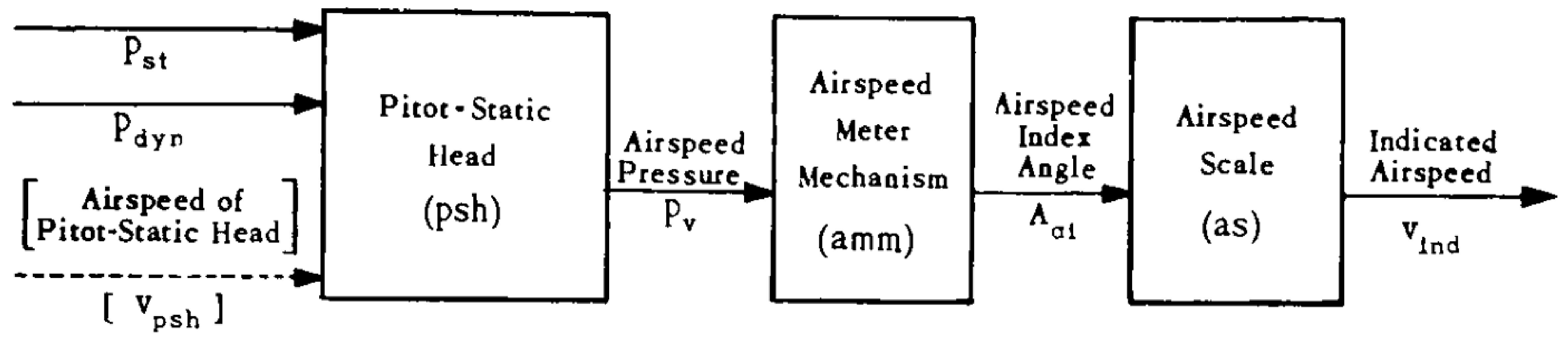
[*] This relationship is modified by a series of correction terms if compressibility effects are considered. See, The Elements of Aerofoil and Airscrew Theory by Glauert (19).



a) Schematic diagram of pitot-static airspeed meter



Note

V_{psh} is the equivalent input corresponding to P_{st} and P_{stgn} as the actual inputs.

b) Functional diagram for pitot-static airspeed meter

From Bernouilli's law for incompressible fluids

$$P_{stgn} - P_{st} = \frac{1}{2} \rho_{air} V_{psh}^2$$

By operation of pitot-static head (psh)

$$P_v = P_{stgn} - P_{st}$$

By operation of airspeed meter mechanism (amm)

$$A_{ai} = S_{(amm)[p;A]} P_v$$

By operation of airspeed scale

$$V_{ind} = S_{(as)[A;v]} A_{ai}$$

c) Characteristics for operating components of pitot-static airspeed meter

Fig. 5-3. Diagrams and operating data for pitot-static airspeed meter.

5-1b is that the indicated airspeed must be equal to the pitot-static head airspeed when the air density has a particular calibration value. This means that the meter reading will always be incorrect when the air density differs from the calibration density.

5.15. For present purposes the important design specification is that the indicated airspeed scale must be uniformly divided so that the airspeed index makes one complete revolution for each* 100 miles per hour change in airspeed.

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* In practice, this requirement would not be applied for airspeeds less than about 50 miles per hour, because flight by craft capable of high speed is not ordinarily possible below 50 miles per hour. From the standpoint of the mechanism, a stop would be used to eliminate index positions below 50 miles per hour.

This implies that the meter mechanism would be designed to permit more than one complete revolution of the index by means of features not shown in Fig. 5-3a. The absence of these features in the diagram is not important for the present discussion, which is concerned only with the method for describing meter performance.

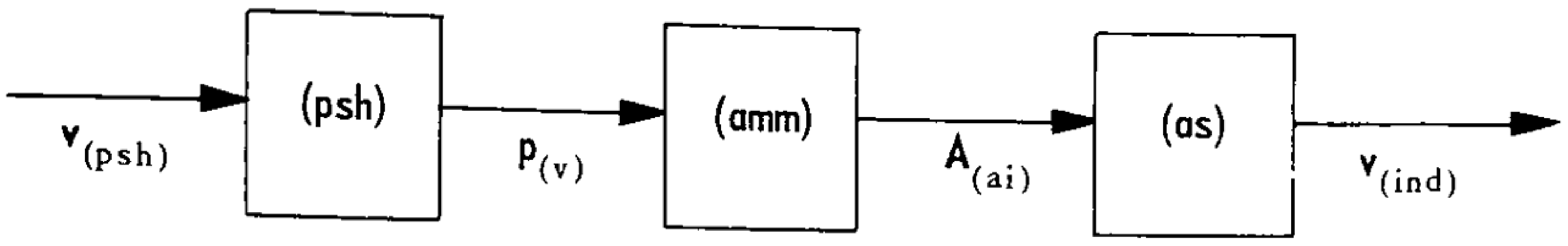
IDEAL AIRSPEED METER PERFORMANCE

5.16. Under the design specifications of Design Performance Summary 5-1b, an ideal airspeed meter would produce airspeed readings equal to the pitot-static head airspeed under calibration conditions, i.e., when the air density is equal to the calibration air density. This performance must be achieved in spite of the fact that the airspeed pressure, which is the output pressure from the pitot-static head, is proportional to the square of the pitot-static head airspeed. In order to have index angle increments directly proportional to airspeed increments, the mechanism of an ideal meter must have a sensitivity that varies properly with the pressure input.

5.17. Equation (4) of Design Performance Summary 5-1b is the airspeed meter performance equation based on the component performance information listed in Design Performance Summary 5-1a. Equation (5) of this summary is similar to Eq. (4), except that the airspeed pressure – airspeed relationship of Eq. (1) is used to replace one of the airspeed symbols by an expression involving the airspeed pressure and the calibration air density. In order for the indicated airspeed to be equal to the pitot-static head airspeed, it is necessary for the expression in the square bracket of Eq. (5) to be made unity by design of the meter. Under the assumption of constant scale sensitivity, as required by the design requirement of a uniformly divided scale, and with calibration conditions existing, Eq. (6) of the summary is an expression for the pressure – angle sensitivity of the ideal airspeed meter. This expression shows that the ideal mechanism sensitivity must be inversely proportional to the square root of the calibration air density, inversely proportional to the airspeed scale sensitivity (which is constant by specification), and inversely proportional to the square root of the airspeed pressure.

ILLUSTRATIVE AIRSPEED METER PERFORMANCE RATIO

5.18. In describing the performance of an actual airspeed meter intended to meet the performance specifications of Design Performance Summary 5-1b, it is logical to select ideal performance as the standard of comparison. Because the pitot-static head performance is directly controlled by a law of nature, while the scale sensitivity is fixed by design specification, the only component of the airspeed meter available for achieving design specifications is the meter mechanism. The essential characteristic of the meter mechanism is its pressure-angle sensitivity.



(psh)

$$p_v = \frac{1}{2}\rho_{(air)}v^2_{(psh)} \tag{1}$$

Note: This equation is an approximation that is valid only for airspeed ranges in which compressibility effects are negligible. The procedure illustrated here applies without modification if the general airspeed equation is used.

(amm)

$$A_{(ai)} = S_{(amm)[p;A]}p_v \tag{2}$$

(as)

$$v_{(ind)} = S_{(as)[A;v]}A_{(ai)} \tag{3}$$

a) Airspeed meter component characteristics

$v_{(ind)} = v_{(psh)}$ when $\rho_{(air)} = \rho_{(air)(calib)} = 0.0012255$ gram per cm^3 (within tolerance limits)

One complete revolution of the index must correspond to 100 miles per hour change in the indicated airspeed with uniform scale divisions; i.e.,

$$S_{(as)[A;v]} = \text{constant} = \frac{100}{360} = \frac{1}{3.6} \text{ mph per degree}$$

Tolerance limits are not concerned in the present discussion and for this reason are not specified here.

b) Design specifications

Under calibration conditions (i.e., $\rho_{(air)} = \rho_{(air)(calib)}$) the airspeed meter performance equation is

$$v_{(ind)} = [S_{(as)[A;v]}S_{(amm)[p;A]}]\frac{1}{2}\rho_{(air)(calib)}v^2_{(psh)} \tag{4}$$

Operation of the pitot-static head relates $v_{(psh)}$ to p_v so that the airspeed meter performance equation may be written

$$v_{(ind)} = \left[S_{(as)[A;v]}S_{(amm)[p;A]}\sqrt{\frac{\rho_{(air)(calib)}}{2}}\sqrt{p_v}\right]v_{(psh)} \tag{5}$$

Because p_v is the input for the airspeed meter mechanism, it is possible to make $S_{(amm)[p;A]}$ a function of positions of parts in the mechanism, which in turn are fixed by the index position $A_{(ai)}$. In any satisfactory meter $A_{(ai)}$ depends upon p_v, so $S_{(amm)[p;A]}$ may be considered as dependent on p_v. When calibration conditions exist, an ideal airspeed meter will operate so that

$$v_{(ind)} = v_{(psh)}$$

This result is achieved if the airspeed meter mechanism performance equation has the form

$$S_{(amm)[p;A](ideal)} = \frac{1}{S_{(as)[A;v]}}\sqrt{\frac{2}{\rho_{(air)(calib)}}}\frac{1}{\sqrt{p_v}} \tag{6}$$

c) Mechanism performance equations

Design Performance Summary 5-1. Performance equations for a uniform scale airspeed meter.

5.19. The importance of the pressure — angle mechanism sensitivity in the performance of the illustrative airspeed meter suggests that a performance ratio for the meter be formed on the basis of this quantity. In order for the performance ratio to be substantially unity in the region of satisfactory operation, the reference sensitivity must be chosen as equal to the ideal mechanism sensitivity. Because this ideal sensitivity is not constant, but must be a function of the input pressure to the meter mechanism, it is necessary to use a variable reference sensitivity in forming the performance ratio. When this requirement is accepted the reference sensitivity will be a function of the airspeed pressure, so it is logical to define the pressure level ratio as the ratio of mechanism input pressure to a selected constant pressure, and to use this ratio as the operating level ratio in making performance ratio plots.

LOGARITHMIC SCALE PERFORMANCE RATIO PLOTS WITH VARIABLE REFERENCE SENSITIVITY

5.20. Variable reference sensitivities may in certain cases be represented by simple curves if log-log coordinates rather than linear coordinates are used. This property is an advantage for logarithmic scales over and above the range extension characteristic already discussed in connection with Fig. 5-2. Derivation Summary 5-1 outlines the background for forming dimensionless ratios suitable for use in describing the performance of the airspeed meter shown in Fig. 5-3.

5.21. Equation (1) of Derivation Summary 5-1 is formed when the airspeed meter mechanism performance equation (6) of Design Performance Summary 5-1 is both multiplied and divided by an arbitrarily chosen constant reference airspeed pressure. The resulting form of the equation shows that the ideal pressure — angle sensitivity is equal to a constant multiplied by the reciprocal of the square root of the pressure level ratio. By definition the pressure level ratio is equal to the ratio of the airspeed pressure to the reference airspeed pressure. Under this definition the constant multiplying factor for the pressure level term in the expression for the ideal mechanism sensitivity is just the ideal mechanism sensitivity when the pressure level ratio is unity. Using this sensitivity as the reference leads to the expression of Eq. (4) of Derivation Summary 5-1 for the airspeed meter mechanism pressure — angle ideal sensitivity — reference sensitivity ratio. Equation (5) of Derivation Summary 5-1 shows that the ideal sensitivity — reference sensitivity ratio is equal to the pressure level ratio raised to the minus one-half power. This relationship would be a curve on a plot with linear scales but is a straight line with a slope of minus one-half when plotted logarithmically. Equation (6) similarly defines the mechanism sensitivity — reference sensitivity ratio as the ratio of the mechanism sensitivity to the ideal reference sensitivity of Eq. (3).

5.22. Equation (7) of Derivation Summary 5-1 defines the airspeed meter mechanism pressure — angle sensitivity performance ratio as the ratio between

Multiply and divide the airspeed meter mechanism performance equation by $p_{v(ref)}$, a reference airspeed pressure taken as constant at an arbitrary level. This gives

$$S_{(amm)[p;A](ideal)} = \frac{1}{S_{(as)[A;V]}} \sqrt{\frac{2}{\rho_{(air)(calib)}}} \; \frac{1}{\sqrt{p_{v(ref)}}} \left[\frac{p_{v(ref)}}{p_v}\right]^{1/2} \tag{1}$$

Define

$$PLR = \frac{p_v}{p_{v(ref)}} \equiv \text{Pressure level ratio} \tag{2}$$

$$S_{(amm)[p;A](ideal)[PLR=1]} = \frac{1}{S_{(as)[A;V]}} \sqrt{\frac{2}{\rho_{(air)(calib)}}} \; \frac{1}{\sqrt{p_{v(ref)}}} \tag{3}$$

$$[S(RS)R]_{(amm)[p;A](ideal)} = \frac{S_{(amm)[p;A](ideal)}}{S_{(amm)[p;A](ideal)[PLR=1]}} \tag{4}$$

$\equiv$ Airspeed meter mechanism pressure – angle ideal sensitivity – reference sensitivity ratio

$$[S(RS)R]_{(amm)[p;A](ideal)} = (PLR)^{-1/2} \tag{5}$$

$$[S(RS)R]_{(amm)[p;A]} = \frac{S_{(amm)[p;A]}}{S_{(amm)[p;A](ideal)[PLR=1]}} \tag{6}$$

a) Definitions of mechanism sensitivity and sensitivity-reference sensitivity ratio

$$(PR)_{(amm)[S_{[p;A]}]} = \frac{S_{(amm)[p;A]}}{S_{(amm)[p;A](ref)}} \tag{7}$$

$\equiv$ Airspeed meter mechanism pressure – angle sensitivity performance ratio

$S_{(amm)[p;A](ref)} \equiv$ Airspeed meter mechanism pressure – angle reference sensitivity

This reference sensitivity may be constant or varying in an arbitrary way. It is generally chosen for convenience in the representation of the operating system under consideration. For the present case choose

$$S_{(amm)[p;A](ref)} = S_{(amm)[p;A](ideal)} \tag{8}$$

From Eqs. (6), (7) and (8)

$$(PR)_{(amm)[S_{[p;A]}]} = \frac{[S(RS)R]_{(amm)[p;A]}}{[S(RS)R]_{(amm)[p;A](ideal)}} \tag{9}$$

Taking logarithms of both sides of Eq. (9) and rearranging terms

$$\log [S(RS)R]_{(amm)[p;A]} = \log [S(RS)R]_{(amm)[p;A](ideal)} + \log (PR)_{(amm)[S_{[p;A]}]} \tag{10}$$

or

$$\log (PR)_{(amm)[S_{[p;A]}]} = \log [S(RS)R]_{(amm)[p;A]} - \log [S(RS)R]_{(amm)[p;A](ideal)} \tag{11}$$

b) Derivation of the performance ratio equation with the ideal sensitivity taken as the reference sensitivity

Derivation Summary 5-1. Performance ratio equation for airspeed meter mechanism.

the actual sensitivity (represented by the sensitivity symbol already described with the subscript (ideal) omitted) and the ideal sensitivity.

5.23. Equation (9) of Derivation Summary 5-1 shows that the mechanism performance ratio is equal to the ratio of the sensitivity – reference sensitivity ratio to the ideal sensitivity – reference sensitivity ratio. Equation (10) of Derivation Summary 5-1 gives the corresponding logarithmic form of this relationship, which shows that the logarithm of the actual sensitivity – reference sensitivity ratio is equal to the logarithm of the ideal sensitivity – reference sensitivity ratio plus the logarithm of the performance ratio. Equation (11) of Derivation Summary 5-1 shows this equation solved for the logarithm of the performance ratio.

SYSTEMS OF LOGARITHMIC COORDINATES

5.24. Figure 5-4 shows the relationships of Derivation Summary 5-1 plotted with logarithmic scales on an illustrative performance-describing plot* for the airspeed meter of Fig. 5-3. The field of the plot itself is divided logarithmically for both the ordinate and abscissa scales. Outside of the field, lines are placed with twenty uniform divisions for one logarithmic cycle. These uniform divisions may be used to demonstrate the three systems of commonly-used logarithmic coordinates:

1) One cycle of logarithms to the base 10 is one logarithmic ratio unit – (symbol loru). A loru abscissa scale is shown in Fig. 5-4.
2) One cycle of logarithms to the base 10 is twenty decibels – (symbol db). A decibel ordinate scale appears in Fig. 5-4, outside the loru scale.
3) Three tenths (approximately) of one loru (6 db) is equal to one octave, which is a logarithmic system with 2 as the base. An abscissa scale divided into octaves is placed below the loru scale.

5.25. Each of the four scale unit systems shown in Fig. 5-4 is in common use. In practice it is unimportant which system is employed in a given situation so long as the proper conversions are used between systems.

PERFORMANCE RATIO AND PERFORMANCE RATIO DEVIATION PLOTS ON LOGARITHMIC COORDINATES

5.26. In Fig. 5-4 the ideal sensitivity – reference sensitivity ratio is shown as a heavy dashed straight line sloping downward to the right with a slope of minus one-half. This line passes the unity ordinate level on the direct scale for a pressure level ratio of unity, i.e., at the point the input pressure for the mechanism becomes equal to the reference pressure. In order to interpret performance plots similar to those of Fig. 5-4, it is only necessary to specify the

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* The information plotted in Fig. 5-4 does not represent typical performance for any specific airspeed meter. It is arbitrarily chosen for demonstration purposes without regard to whether or not it describes achievable performance.

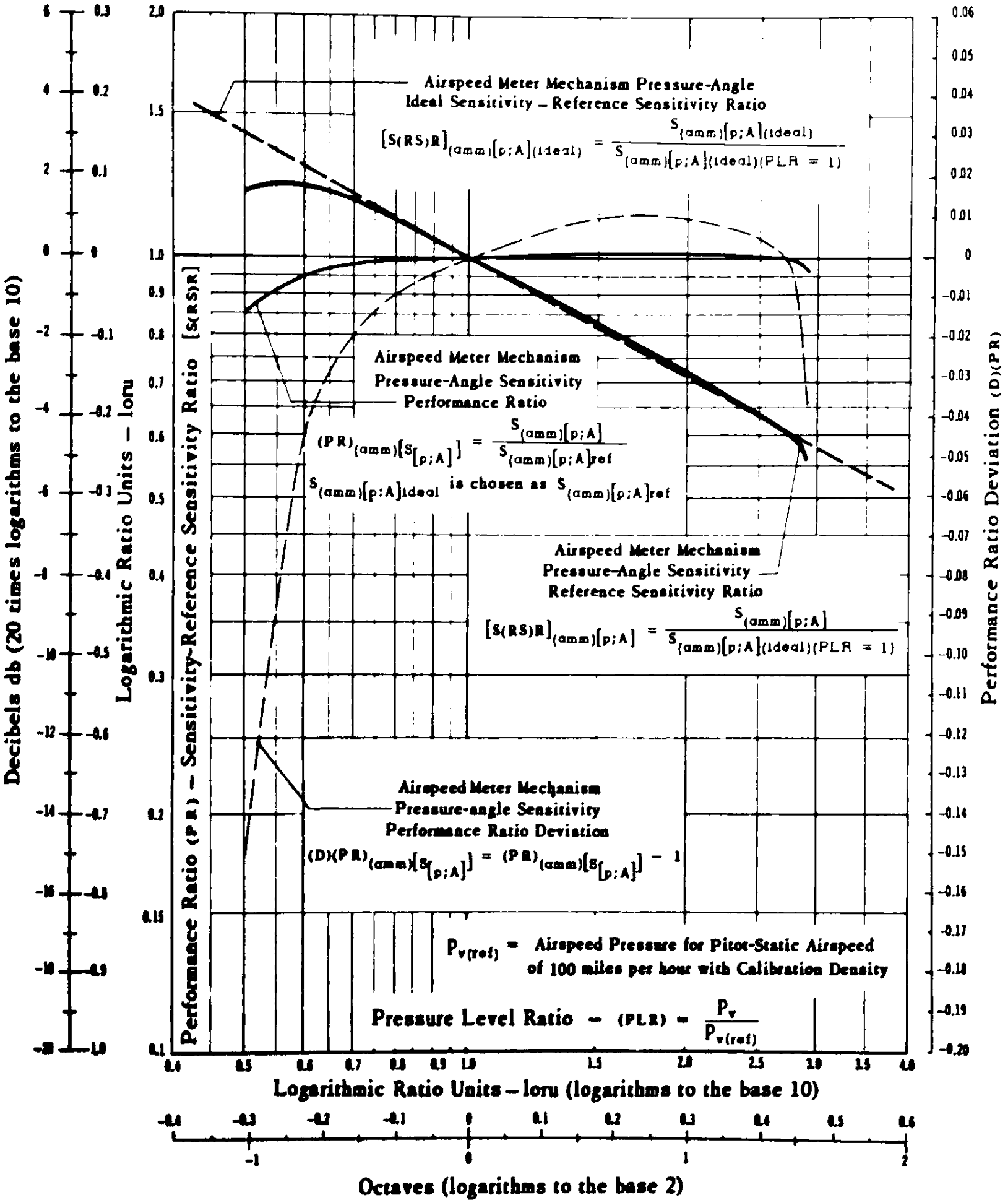

Fig. 5-4. Performance ratio and performance ratio deviation for pitot-static airspeed mechanism.

reference pressure level. In Fig. 5-4 the reference pressure was chosen as the airspeed pressure corresponding to one hundred miles per hour with air density equal to the calibration air density.

5.27. The airspeed meter mechanism pressure – angle sensitivity – reference sensitivity ratio curve shown by the heavy full line of Fig. 5-4 represents the actual airspeed meter performance. In practice this curve would be derived from measured data and located on the pressure level ratio scale from a knowledge of the reference pressure. Because of the relationship expressed in Eq. (11) of Derivation Summary 5-1, all the points for the performance ratio curve may be taken as the distance from the ideal sensitivity – reference sensitivity ratio curve to the actual sensitivity – reference sensitivity ratio curve stepped off from the unity line of the ordinate scale in terms of numerical values of the performance ratio. (Note that this line is identical with the zero line of the logarithmic unit ordinate scale.) In Fig. 5-4 the performance ratio curve is shown by a full line of medium weight.

5.28. The difficulty of reading small differences of the performance ratio from unity is overcome by using the twenty uniform divisions per logarithmic cycle ordinate scale in making a direct number plot of the performance ratio deviation. The thin dashed line of Fig. 5-4 shows the performance ratio deviation plotted to a scale that makes the space of one logarithmic cycle equal to two-tenths of one deviation unit. The choice of an expanded linear scale for deviations means that deviation curves represent a magnified picture of the differences of the performance ratio curve from unity. The magnification means that data for deviation curves will generally have to be derived directly by computation from experimental data instead of being found by the graphical process outlined above. In fact, with the background outlined above clearly understood, all the essential data on the static performance of an operating component may be summarized in deviation curves for properly selected performance ratios and the necessary associated information on definitions and reference quantities.

5.29. The detailed discussion given above of the procedure for representing the static performance of a hypothetical airspeed meter is certainly not justified by the importance of the example itself. However, the effort is not wasted because it has produced a model for demonstrating concepts and procedures that may be used in many situations. In practice, static performance may be represented very simply by a single performance deviation curve, or it may include plots similar to those of Fig. 5-4 and background information like that contained in Fig. 5-3, Design Performance Summary 5-1, and Derivation Summary 5-1. The procedure in any given case must be determined by the engineer's good judgment based on his knowledge of any given situation.

CHAPTER 6

DYNAMIC PERFORMANCE

DYNAMIC CONDITIONS: THE FORCING FUNCTION

6.1. Dynamic conditions for a given operating component or system occur when the relationship between the input and the output is different from the relationship that would exist under static or quasi-static conditions for the same component or system. Under this definition it is logical in any particular situation to use the output variation that would occur if dynamic effects were absent, that is, to use the quasi-static output variation, as the reference level for describing the effects of operating system* performance on the actual output. Because quasi-static output variations correspond directly to the input changes that force the actual output changes, the quasi-static output is also called the forcing output. When the forcing output is described in mathematical terms, or by graphical means, the represented variation with time is the output forcing function or, where no misunderstanding is possible, the forcing function. The output variation corresponding to the output forcing function is the output response, or the response, of an operating system. The input change corresponding to the output forcing function is the input forcing function.

DYNAMIC ERROR AND DYNAMIC CORRECTION

6.2. In Fig. 6-1 the dashed line shows a possible quasi-static output variation for an operating component. The full-line curve in the same figure represents the corresponding output response. For a typical time instant, t, the dynamic error is the difference between the output response level at t, and the forcing function point for the same instant. This definition means that the dynamic correction at t, which is defined as the negative of the error, is added algebraically to the response point in order to find the corresponding forcing function point.

DYNAMIC RESPONSE DELAY

6.3. Under dynamic conditions the output response level at any instant may be equal to the output forcing function level at some other instant of time. By definition the dynamic response delay is the time difference between the instant a given output response level exists and some instant at which the output forcing function has this level. The concept of dynamic response delay is illustrated in Fig. 6-1. It is to be noted that the response delay is positive when a given output response level occurs after the output forcing function has passed through this level. That is, the dynamic response delay correction, which is by definition the negative of the dynamic response delay, becomes negative when the corresponding error is positive. This negative sign shows that a time interval should be subtracted from the instant of occurrence of a given output level in order to find an instant at which the forcing function is at this level.

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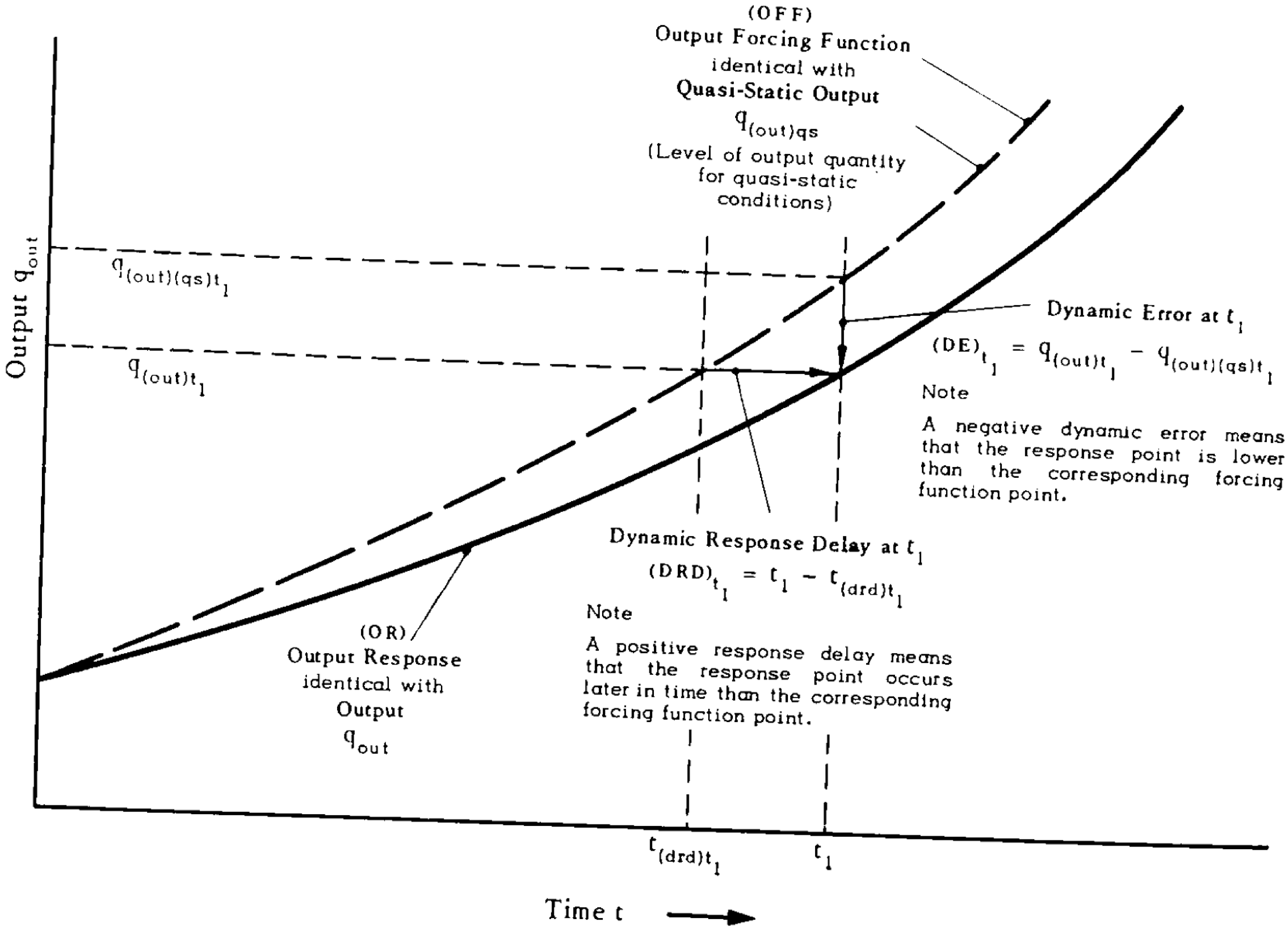
* The words "operating system" will be generally used to include both operating components and operating systems. This is logical in that a component is essentially a system with a single part.



(OFF)
Output Forcing Function identical with Quasi-Static Output
$q_{(out)qs}$
(Level of output quantity for quasi-static conditions)

Dynamic Error at t_1
$(DE)_{t_1} = q_{(out)t_1} - q_{(out)(qs)t_1}$

Note

A negative dynamic error means that the response point is lower than the corresponding forcing function point.

Dynamic Response Delay at t_1
$(DRD)_{t_1} = t_1 - t_{(drd)t_1}$

Note

A positive response delay means that the response point occurs later in time than the corresponding forcing function point.

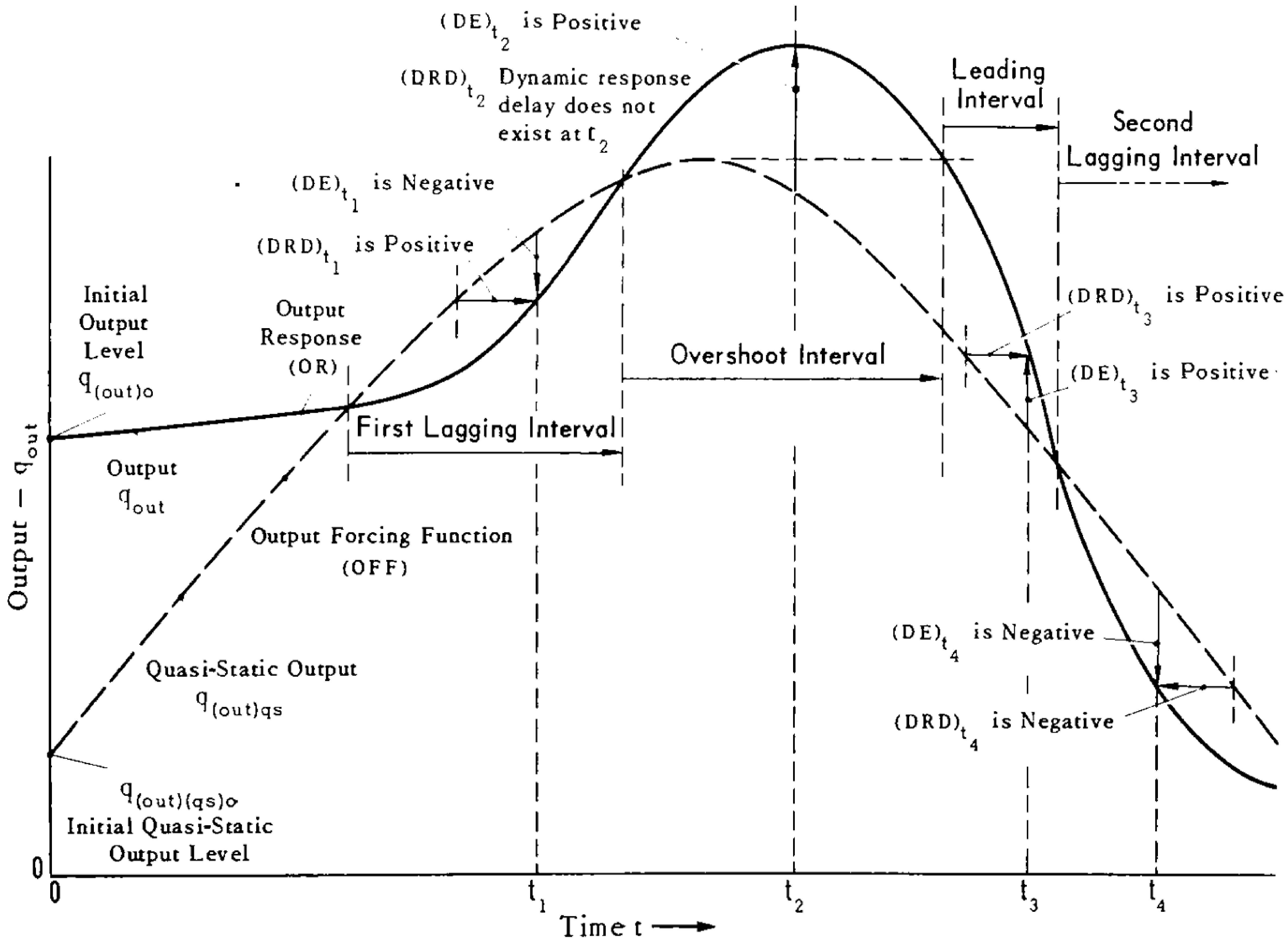
(OR)
Output Response identical with Output
q_{out}

Fig. 6-1. Output response under dynamic conditions.

SPECIAL CASES OF RESPONSE DELAY AND DYNAMIC ERROR

6.4. Three possible situations that may occur under dynamic conditions are illustrated by the curves of Fig. 6-2. The dashed line represents the output forcing function. The corresponding output response is shown by the full-line curve. The forcing function is assumed to start at an initial level for the instant chosen for time equal to zero. The initial response level is assumed to be above the initial forcing function level. As the forcing function undergoes a change with the general shape of a half sinusoid, the response is first "ahead" because of the initial condition chosen, then "lags" behind until the forcing function peak is approached, when it "overshoots," and finally lags again for a time interval near the right-hand side of the figure.

6.5. At a typical point, t_1, in the first "lagging" interval of Fig. 6-2, the dynamic error is negative and the dynamic response delay is positive. At t_2, during the overshoot period, the dynamic error is positive when the response is actually above the forcing function. Because the forcing function never actually reaches the response level during part of the overshoot interval, the concept of dynamic response delay is without meaning in this region and cannot be given for the time instant t_2. In Fig. 6-2 a "leading" interval follows the overshoot interval. At t_3 in this period the dynamic error is positive, and the dynamic response delay is also positive. At the typical point t_4 of the second lagging interval the dynamic error is negative, while the dynamic response delay is also negative.



(DE)$_{t_2}$ is Positive
(DRD)$_{t_2}$ Dynamic response delay does not exist at t_2
(DE)$_{t_1}$ is Negative
(DRD)$_{t_1}$ is Positive
Initial Output Level $q_{(out)o}$
Output Response (OR)
First Lagging Interval
Overshoot Interval
Leading Interval
Second Lagging Interval
(DRD)$_{t_3}$ is Positive
(DE)$_{t_3}$ is Positive
Output q_{out}
Output Forcing Function (OFF)
Quasi-Static Output $q_{(out)qs}$
$q_{(out)(qs)o}$ Initial Quasi-Static Output Level
(DE)$_{t_4}$ is Negative
(DRD)$_{t_4}$ is Negative
Output — q_{out}
0
0
t_1
t_2
t_3
t_4
Time t ⟶

Fig. 6-2. Dynamic error and response delay in an oscillatory response to a forcing function of general shape.

6.6. The four regions of Fig. 6-2 illustrate several typical situations that may occur under dynamic conditions, and show how the essential information associated with output response may be expressed in terms of dynamic error and dynamic response delay. These concepts will be applied many times in examples that are discussed later.

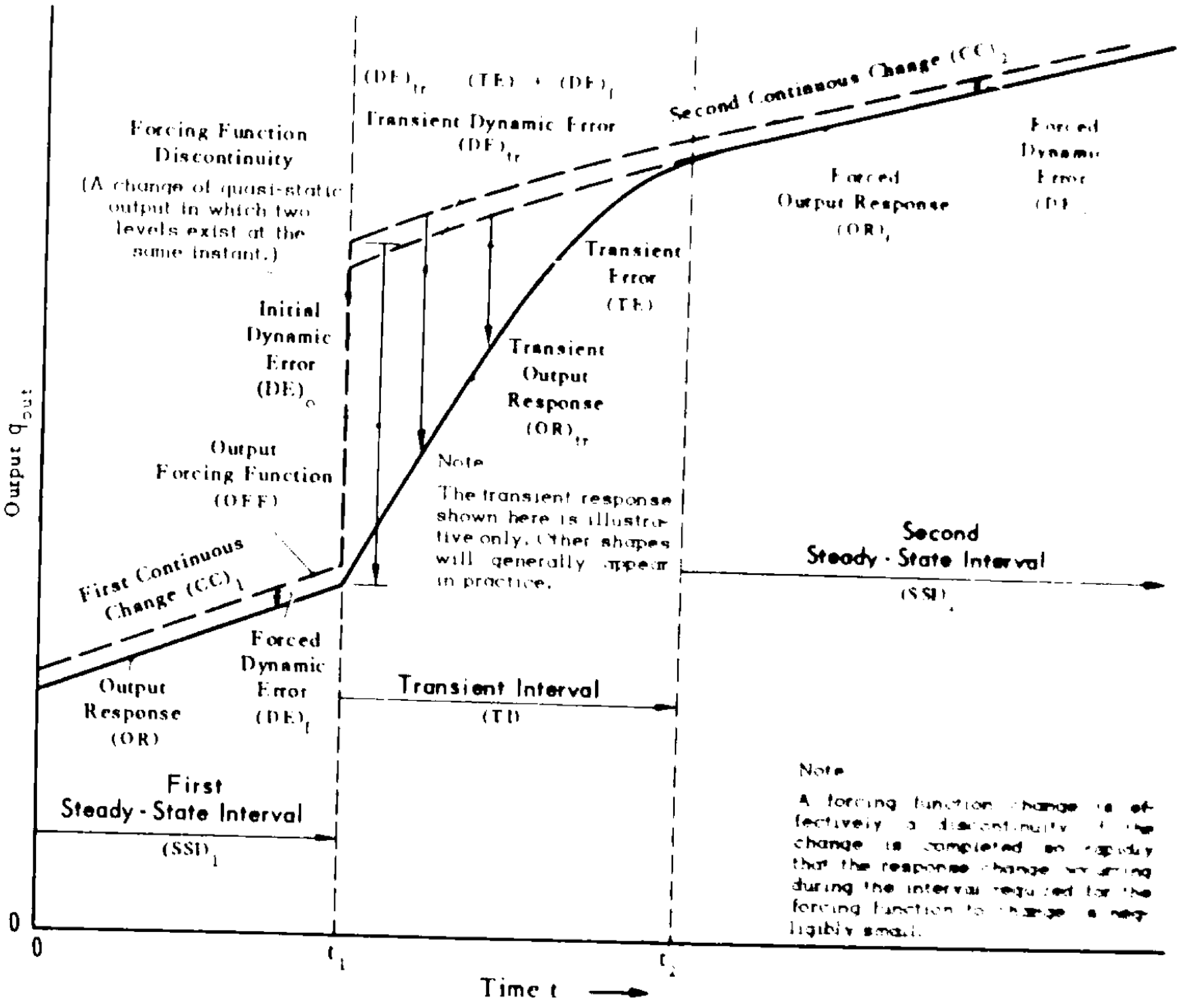
CONTINUOUS AND DISCONTINUOUS FORCING FUNCTION CHANGES

6.7. For theoretical purposes, forcing function changes may be divided into two classes:

(1) Continuous changes — in which the forcing function has a single value and a single time rate of change at each instant.

(2) Discontinuous changes — in which the level of the forcing function changes instantaneously so that it has two values, or two rates of change, at the same instant of time.

From the standpoint of physical facts, discontinuous variations are not possible in nature. However, any forcing function change that takes place so rapidly that it is completed before any significant output response change can occur falls effectively into the class of discontinuous changes for the operating system considered.



Note

The transient response shown here is illustrative only. Other shapes will generally appear in practice.

Note

A forcing function change is effectively a discontinuity if the change is completed so rapidly that the response change occurring during the interval required for the forcing function to change is negligibly small.

Fig. 6-3. Transient dynamic error and forced dynamic error.

STEADY-STATE CONDITIONS

6.8. In Fig. 6-3 a first continuous change in the output forcing function is followed by a discontinuity and then by a second continuous change region. During the first continuous change region the response differs from the forcing function by the forced dynamic error. Because this is an unchanging condition so long as the forcing function undergoes only continuous changes, the time interval during which it exists is a steady-state interval. At time equal to t_1 in the figure, the forcing function changes instantaneously from one level to another. This change is discontinuous in that two levels of the forcing function exist at the same instant. Because it is impossible for any actual operating component to respond instantaneously at the initial instant, the forcing function jumps ahead of the actual output by the amount of the discontinuous change. After the initial instant the output response immediately starts to approach the continuous forcing function that follows the discontinuity.

TRANSIENT CONDITIONS

6.9. During the interval from the discontinuity until the output becomes identical with the forced output response, transient conditions exist. For the purpose of this definition, the forced output response to a given forcing function is the response that exists when a very long time interval has elapsed after the occurrence of a discontinuity.* In the diagram of Fig. 6-3 the corresponding transient interval is the time required for the initial transient dynamic error to become substantially equal to the forced dynamic error for the second continuous change condition. The transient dynamic error may be considered to be the sum of the forced dynamic error and the transient error. The transient error exists because of the discontinuity of the forcing function. The forced dynamic error is that which would appear if there were no discontinuity. For practical purposes the length of the transient interval depends upon the tolerance limit associated with the interpretation of the word "substantially." A solution for this problem is given in the discussion of Fig. 6-8.

TRANSIENT STABILITY AND TRANSIENT INSTABILITY

6.10. In practice it is possible for operating system outputs to move away from, rather than to approach, the forced response after a forcing function discontinuity has occurred. A system which causes its output to approach the forced response as its equilibrium condition after a forcing function discontinuity has transient stability. A system which causes its output to diverge from the forced response after a forcing function discontinuity has transient instability. Response curves illustrating transient stability and transient instability for a discontinuous

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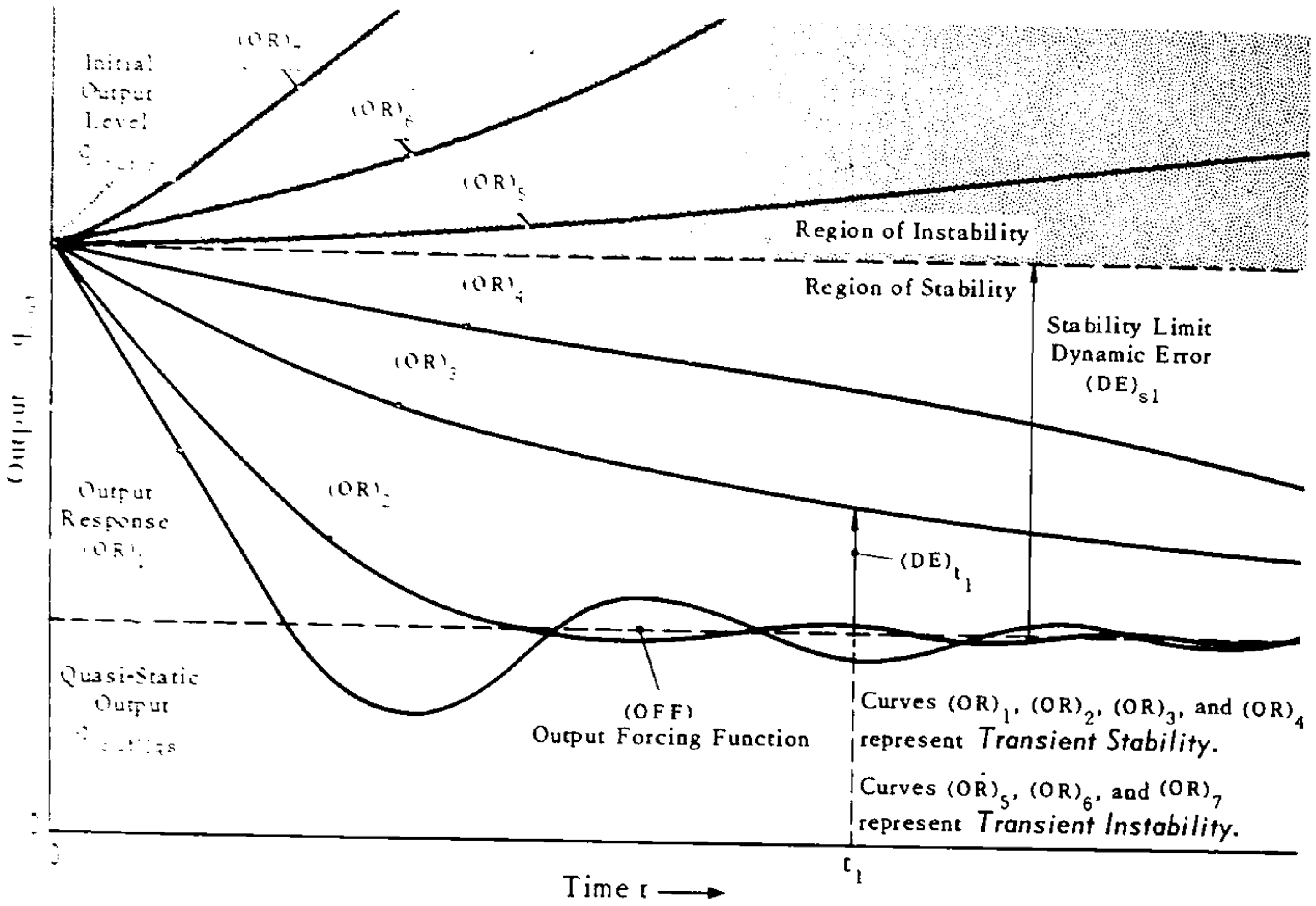
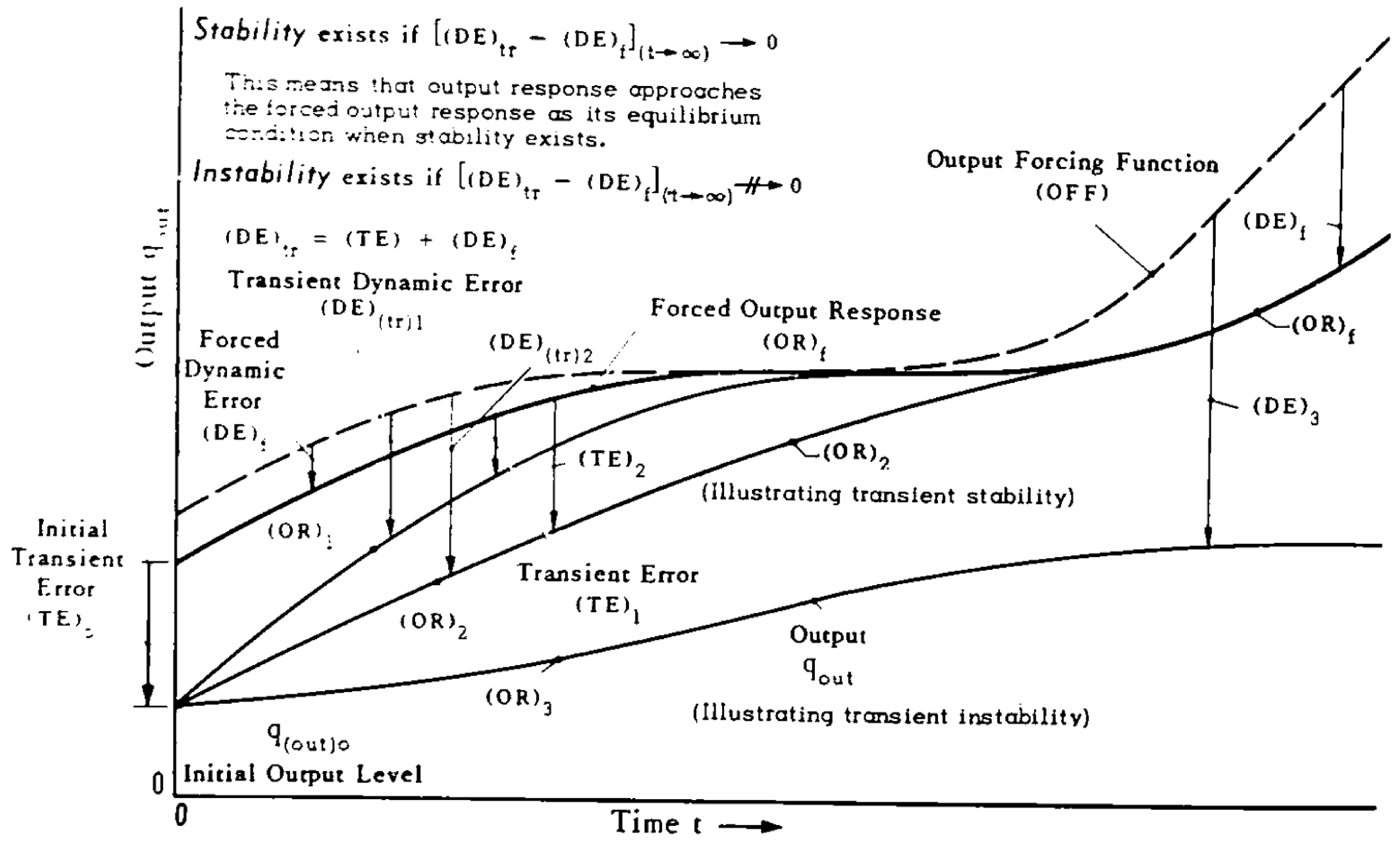
* Transient response and forced response are difficult to define without recourse to mathematical equations. These definitions are discussed in Vol. II.



Curves $(OR)_1$, $(OR)_2$, $(OR)_3$, and $(OR)_4$ represent *Transient Stability*.

Curves $(OR)_5$, $(OR)_6$, and $(OR)_7$ represent *Transient Instability*.

Fig. 6-4. Step function transient stability and instability.



Stability exists if $[(DE)_{tr} - (DE)_f]_{(t \to \infty)} \to 0$

This means that output response approaches the forced output response as its equilibrium condition when stability exists.

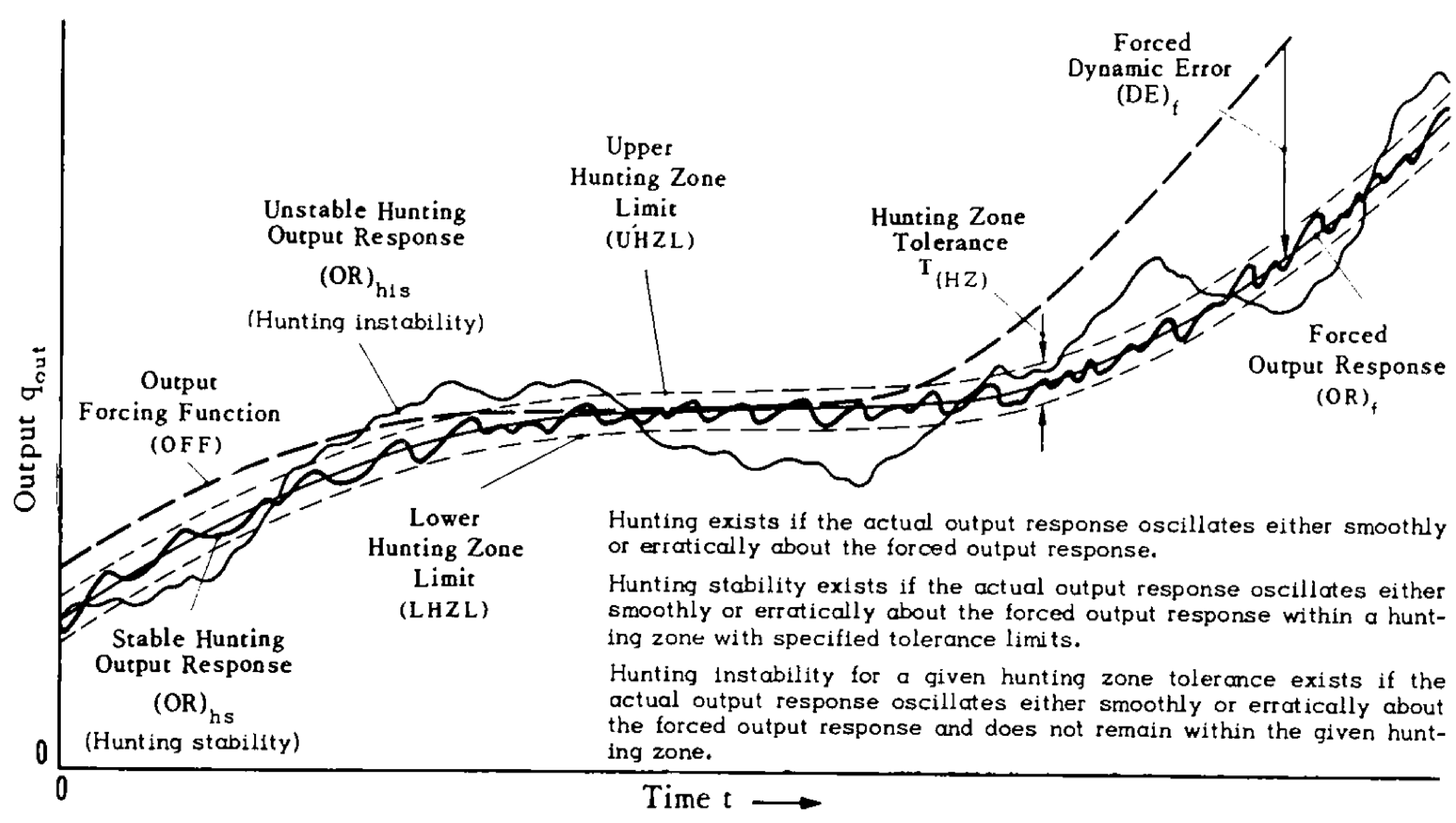
Instability exists if $[(DE)_{tr} - (DE)_f]_{(t \to \infty)} \not\to 0$

$(DE)_{tr} = (TE) + (DE)_f$

Fig. 6-5. Stability and instability for arbitrary forcing function changes.

forcing function change between constant levels are given in Fig. 6-4. Because of the similarity of the forcing function change to a step, the curves of Fig. 6-4 are step function transients. Any output response curve that approaches the constant forcing function level represents transient stability. This is illustrated by the first, second, third, and fourth output response curves of Fig. 6-4. The fifth, sixth, and seventh output response curves move away from the output forcing function level and for this reason illustrate transient instability.

6.11. The concepts of transient stability and transient instability apply not only to step functions but also to discontinuities followed by any arbitrary continuous forcing function changes. The first and second output response curves of Fig. 6-5 illustrate transient stability as they change from an initial transient error to the forced dynamic error for the assumed forcing function. The third output response curve of Fig. 6-5 illustrates transient instability because its dynamic error diverges from the forced dynamic error as time goes on.



Output q_{out}
Output Forcing Function (OFF)
Unstable Hunting Output Response $(OR)_{his}$ (Hunting instability)
Upper Hunting Zone Limit (UHZL)
Hunting Zone Tolerance $T_{(HZ)}$
Forced Dynamic Error $(DE)_f$
Forced Output Response $(OR)_f$
Lower Hunting Zone Limit (LHZL)
Stable Hunting Output Response $(OR)_{hs}$ (Hunting stability)
0
0
Time t

Hunting exists if the actual output response oscillates either smoothly or erratically about the forced output response.

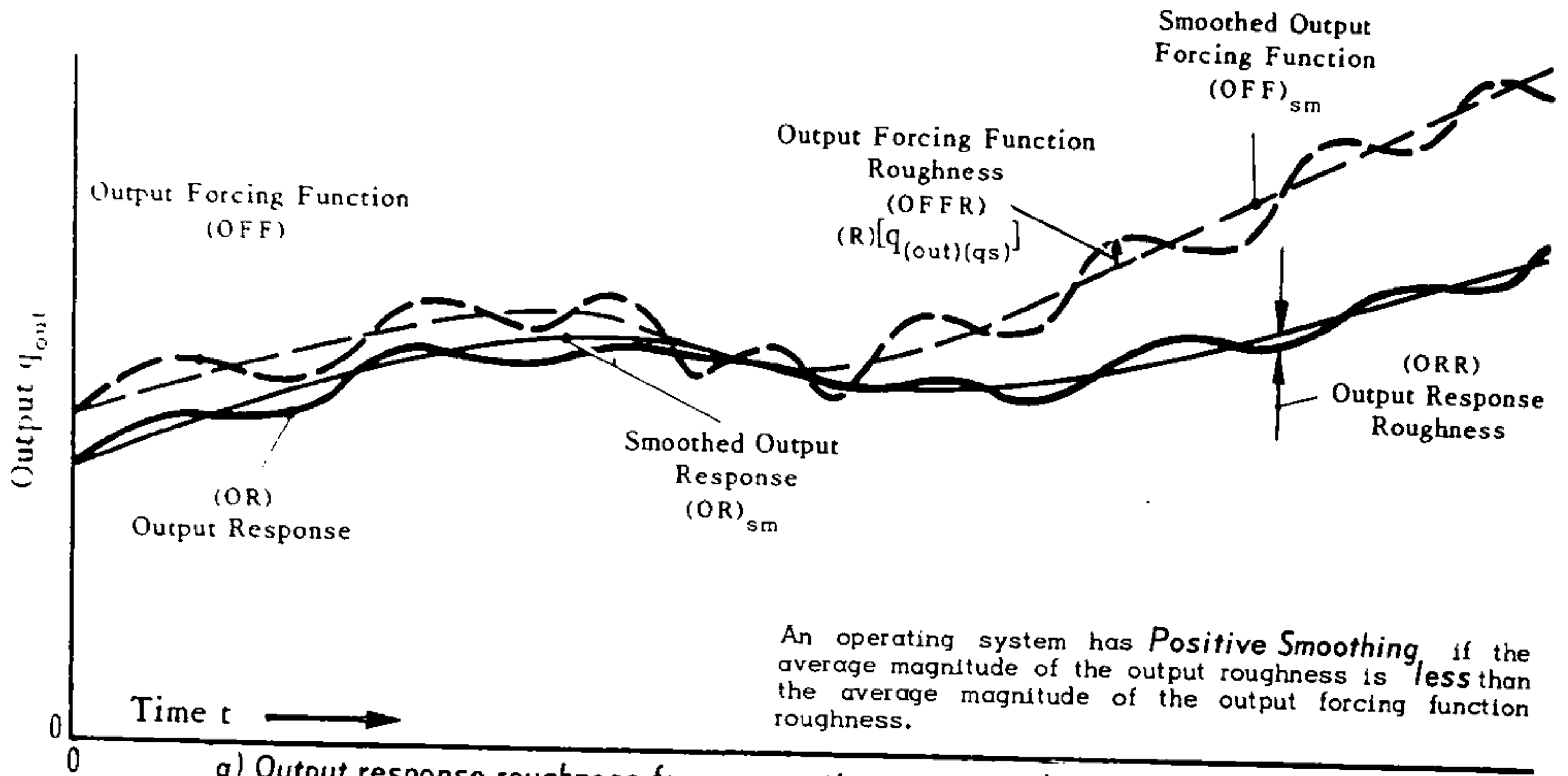
Hunting stability exists if the actual output response oscillates either smoothly or erratically about the forced output response within a hunting zone with specified tolerance limits.

Hunting instability for a given hunting zone tolerance exists if the actual output response oscillates either smoothly or erratically about the forced output response and does not remain within the given hunting zone.

Fig. 6-6. Hunting stability and hunting instability.

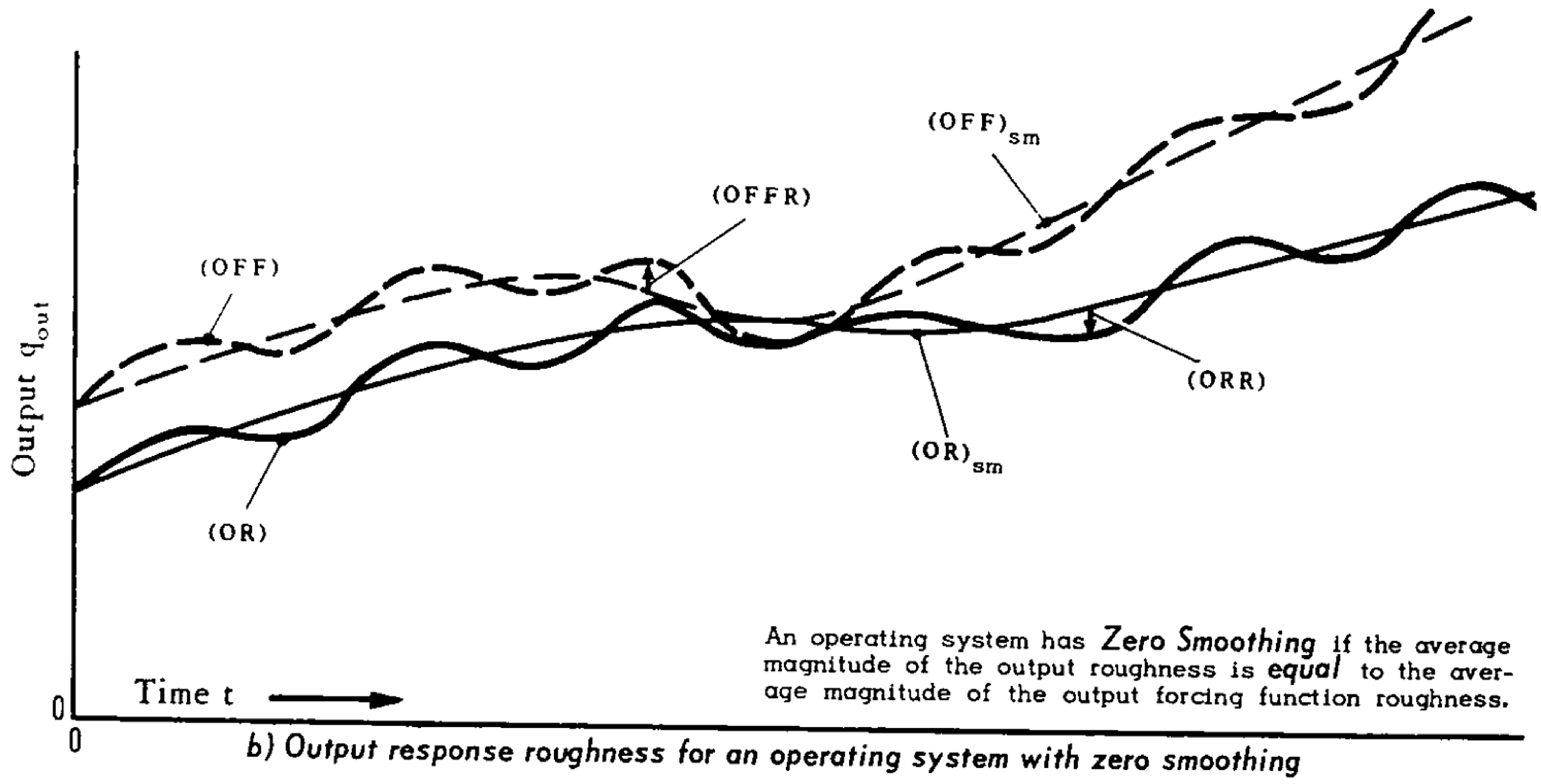
HUNTING STABILITY AND HUNTING INSTABILITY

6.12. Many operating systems are subject to a form of instability that does not fall into the pattern of transient instability. Instability of this kind exists when the output response does not leave the vicinity of the forced output response but hunts more or less erratically about the forced response. The criterion for distinguishing between hunting stability and hunting instability may be a hunting zone tolerance bounded by an upper hunting zone limit and a lower hunting zone limit. A hunting zone tolerance is illustrated in Fig. 6-6, with a stable hunting



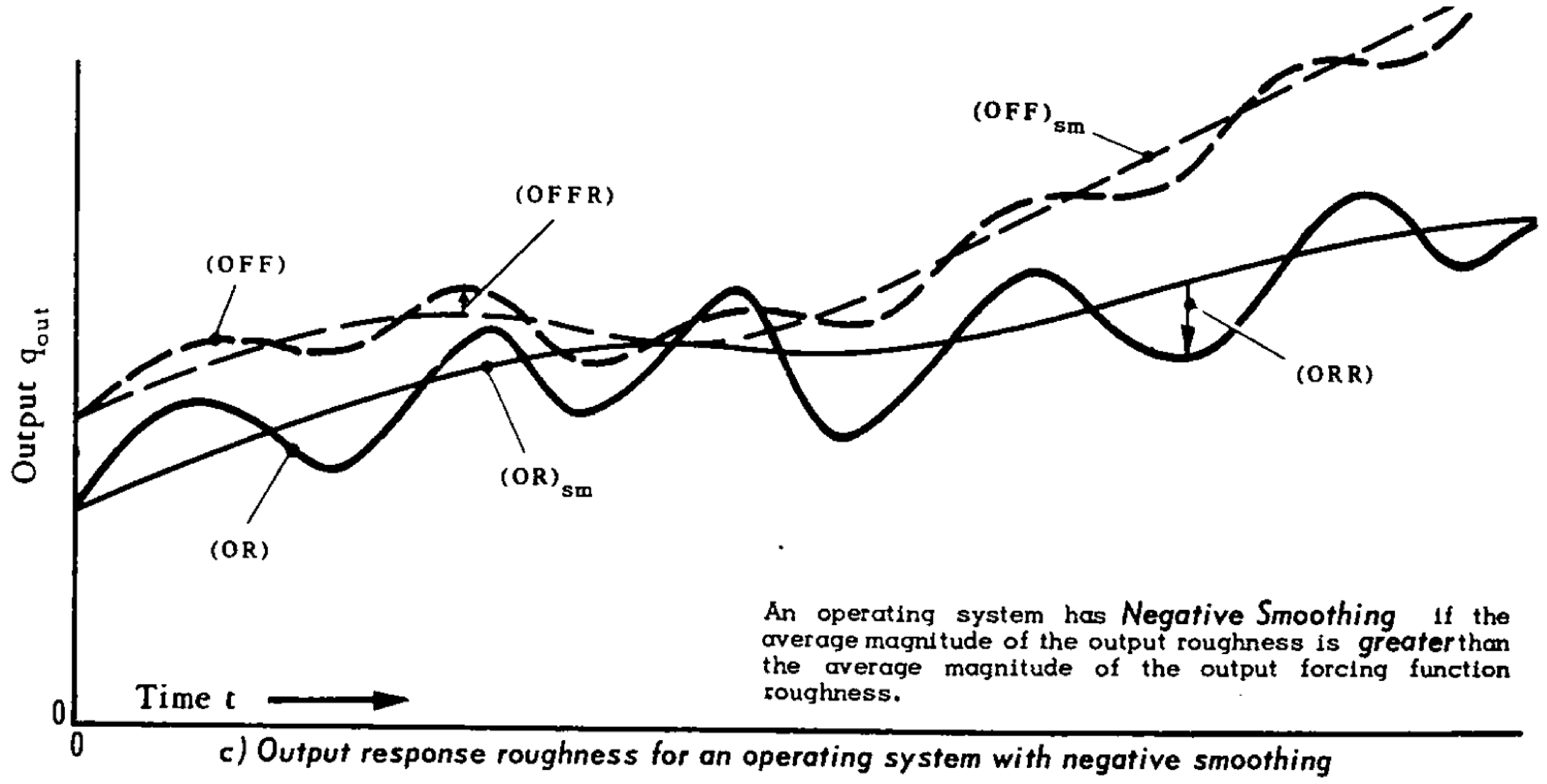
An operating system has *Positive Smoothing* if the average magnitude of the output roughness is *less* than the average magnitude of the output forcing function roughness.

a) Output response roughness for an operating system with positive smoothing



An operating system has *Zero Smoothing* if the average magnitude of the output roughness is *equal* to the average magnitude of the output forcing function roughness.

b) Output response roughness for an operating system with zero smoothing



An operating system has *Negative Smoothing* if the average magnitude of the output roughness is *greater* than the average magnitude of the output forcing function roughness.

c) Output response roughness for an operating system with negative smoothing

Fig. 6-7. Output roughness and smoothing action of an operating system.

output response curve shown as remaining within the hunting zone tolerance and an unstable hunting output response which is outside the hunting zone tolerance during a substantial part of its path.

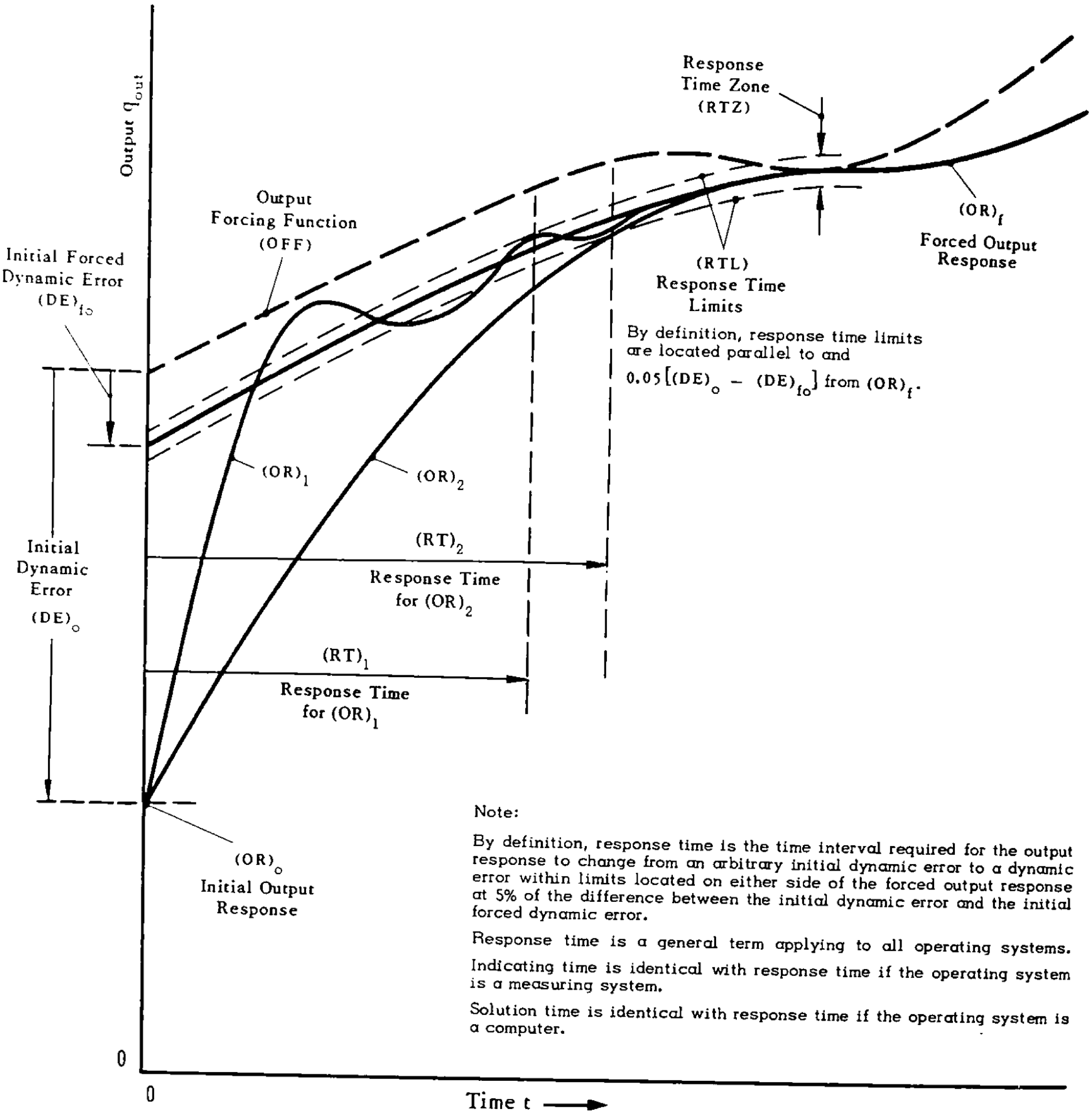
SMOOTHING

6.13. In practice the output forcing function itself may include an erratic component in addition to a general trend. Figure 6-7a shows an output forcing function of this type with an output forcing function roughness superimposed on the smoothed output forcing function. The smoothed output response of Fig. 6-7a is the forced response corresponding to the smoothed output forcing function. The output response roughness is the component corresponding to the output forcing function roughness. When the situation shown in Fig. 6-7a exists, with the average magnitude of the output response roughness less than the average magnitude of the output forcing function roughness, the operating system concerned has positive smoothing. Figure 6-7b illustrates the action of a system with zero smoothing, which operates to give an average output response roughness magnitude equal to the average magnitude of the output forcing function roughness. When the action of an operating system is to produce an average output roughness magnitude greater than the average magnitude of the forcing function roughness, the system has negative smoothing. This situation is illustrated in Fig. 6-7c.

6.14. Various methods of describing the smoothing behavior of an operating system are possible. In general, numbers are chosen to represent smoothing that are negative for negative smoothing, zero for zero smoothing, and positive for positive smoothing. Several methods for describing the smoothing action of an operating system are discussed in later chapters.

RESPONSE TIME

6.15. In practice it is convenient to describe one phase of transient performance in terms of response time. This concept is defined in Fig. 6-8 as the time interval required for the output response to change from an arbitrary initial dynamic error to a substantial coincidence with the forced output response. For purposes of the response time definition, substantial coincidence is taken to mean a zone centered about the forced output response and with a width equal to twice five percent of the difference between the initial dynamic error and the initial forced dynamic error. The magnitude of this zone is chosen so small that it represents substantial completion of transients, while it is still large enough for the slope of the output response to permit satisfactory measurements of response time. When the operating system under discussion is designed for measurement purposes, response time is commonly called indication time. If the system



By definition, response time limits are located parallel to and $0.05[(DE)_o - (DE)_{fo}]$ from $(OR)_f$.

Note:

By definition, response time is the time interval required for the output response to change from an arbitrary initial dynamic error to a dynamic error within limits located on either side of the forced output response at 5% of the difference between the initial dynamic error and the initial forced dynamic error.

Response time is a general term applying to all operating systems.

Indicating time is identical with response time if the operating system is a measuring system.

Solution time is identical with response time if the operating system is a computer.

Fig. 6-8. Definition of response time.

considered is a computer of some type, it is usual to define solution time as being identical with response time.

6.16. Many concepts in addition to those defined above are required to describe the dynamic performance of operating systems. These concepts will appear as they are required for the theoretical developments and problem solutions of this book.

CHAPTER 7

DERIVATION OF PERFORMANCE EQUATIONS

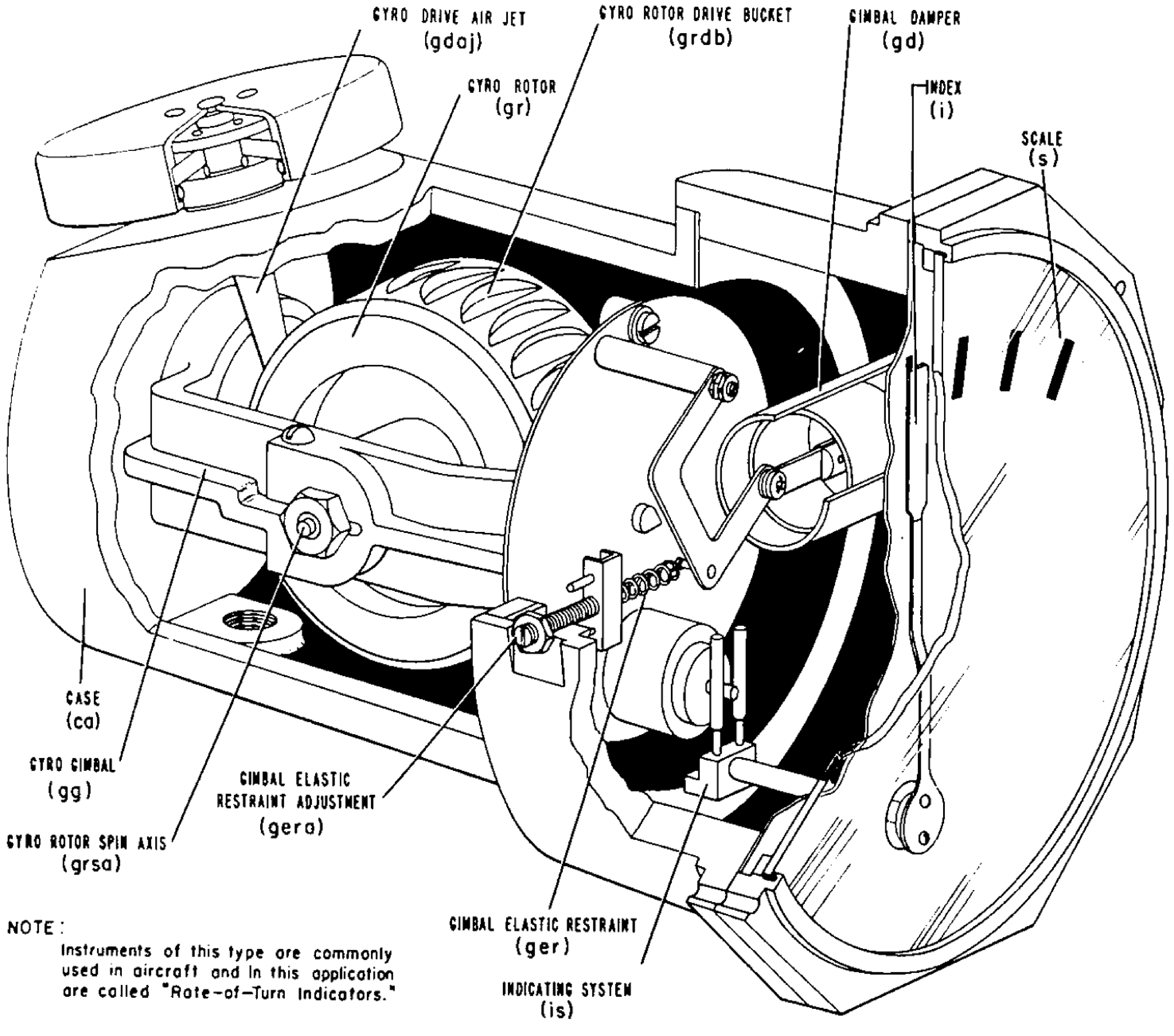
PERFORMANCE EQUATIONS

7.1. Performance-describing concepts like those defined and discussed in the foregoing chapters are necessary for the specification, analysis, design, adjustment, testing, and inspection phases of instrument engineering. In all work of this kind it is of primary importance to stress the quantitative aspect of the problems involved by reducing all performance descriptions to numerical data. When the equipment under consideration exists as a physical item, it is possible to associate numbers with performance-describing concepts by actual measurements. This procedure is not available during the design phases of a development, so that estimates of performance must be based on theoretical considerations; and, in any case, effective engineering always requires that experimental information be considered as part of an information pattern based on theoretical analysis. The first step in any theoretical study is to set up performance equations for the operating systems involved. With these equations available as mathematical forms, the output response corresponding to any given input variation may be found by the procedures of analysis. The performance-describing concepts defined in earlier chapters, and other concepts that will appear in later discussions, are all directly or indirectly associated with performance equations. This means that the performance equation occupies a position of primary importance in the problems of instrument engineering.

7.2. Performance equations are expressions that describe the essential physical actions associated with operating systems. These equations are usually derived by summing the components of some physical quantity associated with a part or parts of the operating system under consideration. For example, the performance equation for the force measuring scale of Fig. 1-6 would be formed by writing an equation for the torque components that act on the scale beam.

DERIVATION OF PERFORMANCE EQUATION FOR GYROSCOPIC RATE-OF-TURN INDICATOR

7.3. The problem of forming the performance equation for any given operating system often imposes special demands on the judgment and background knowledge of the engineer concerned. However, the derivation of equations of this type usually follows a reasonably well-defined pattern that may be illustrated by an example. The gyroscopic rate-of-turn indicator is chosen as an example because an effective performance equation derivation for this instrument must use certain background information not usually known by individuals who may wish to understand the discussion. Situations of this type are so common in engineering that it is worth while to use the rate-of-turn indicator instead of a simpler example for demonstrating the steps of a performance equation derivation. In addition to its value as an illustrative derivation, the rate-of-turn indicator treatment provides information on the performance of a widely-used instrument.



NOTE:
Instruments of this type are commonly used in aircraft and in this application are called "Rate-of-Turn Indicators."

Fig. 7-1. Pictorial diagram for a gyroscopic single-axis angular.velocity indicator.

MECHANICAL FEATURES OF GYROSCOPIC SINGLE-AXIS ANGULAR VELOCITY INDICATOR

7.4. The gyroscopic rate-of-turn indicator is functionally a gyroscopic single-axis (single-degree-of-freedom) angular velocity indicator.* Figure 7-1 is a cutaway mechanical diagram for a gyroscopic single-axis angular velocity indicator. The case of the unit supports the gyro gimbal on bearings which permit rotation about an axis fixed to the case. The gimbal carries the gyro rotor, which is free to rotate with respect to the gimbal about the gyro rotor spin axis. Gyro rotor drive buckets are provided on the outer surface of the rotor, which receives a peripheral force from the gyro drive air jet. The gimbal elastic restraint acts to impose a torque on the gimbal when it has an angular deflection from its position of zero angular velocity equilibrium with respect to the case. The gimbal damper applies a torque to the gimbal that resists rotation with respect to the case by a torque that has a magnitude proportional to the magnitude of the gimbal angular velocity. The indicating system is a lever arrangement that causes the gimbal angular displacement to give a corresponding angular displacement to the index. The scale is placed behind the index to translate index deflections into angular velocity readings.

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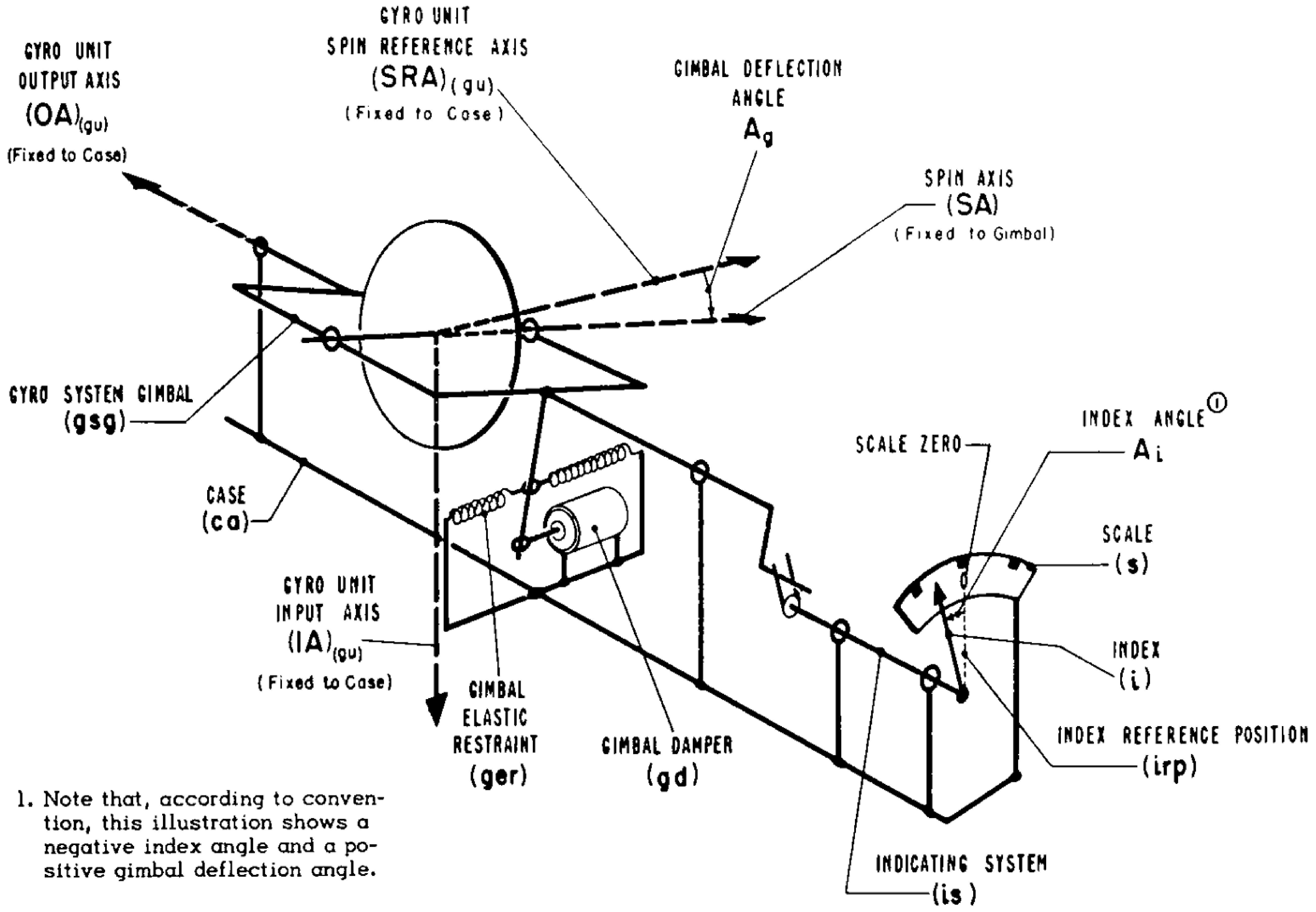
* The instrument is called an angular velocity indicator because under steady-state conditions the position of its index is associated with the angular velocity input to the instrument. The "single-axis" designation means that it is sensitive to angular velocity about a single axis only.



1. Note that, according to convention, this illustration shows a negative index angle and a positive gimbal deflection angle.

Fig. 7-2. Line schematic diagram for a gyroscopic single-axis angular velocity indicator

LINE SCHEMATIC DIAGRAM OF GYROSCOPIC SINGLE-AXIS ANGULAR VELOCITY INDICATOR

7.5. Figure 7-2 is a line schematic diagram for the instrument of Fig. 7-1. Because of its simplicity, the line schematic diagram is well adapted to convey information on special conventions and definitions required for the description of performance for the system represented. Advantage is taken of this fact in Fig. 7-2, which defines the gimbal deflection angle and the index angle. The index angle is measured from the index reference position, which is the position occupied by the index when it is on scale zero. The gimbal deflection angle is measured from the spin reference axis, which is fixed to the case, to the spin axis, which is fixed to the gimbal along the axis about which the rotor spins. The output axis is defined as identical with the axis about which the gimbal rotates with respect to the case. The input axis is at right angles to the output axis and the spin reference axis. The positive directions of these three axes are indicated on Fig. 7-2 by arrowheads on the dashed lines representing the axes themselves.

LINE SCHEMATIC – FUNCTIONAL DIAGRAM OF GYROSCOPIC SINGLE-AXIS ANGULAR VELOCITY INDICATOR

7.6. Figure 7-3 is a line schematic – functional diagram for the instrument of Fig. 7-1. In this diagram, the various operating components are shown with only enough detail for easy identification, and are enclosed in dashed-line boxes connected by arrows representing the physical quantities that are transferred between components. In this figure the input quantity that acts on the gyroscopic system is the input angular velocity. This quantity is by definition the angular velocity of the gyro unit case about the input axis.

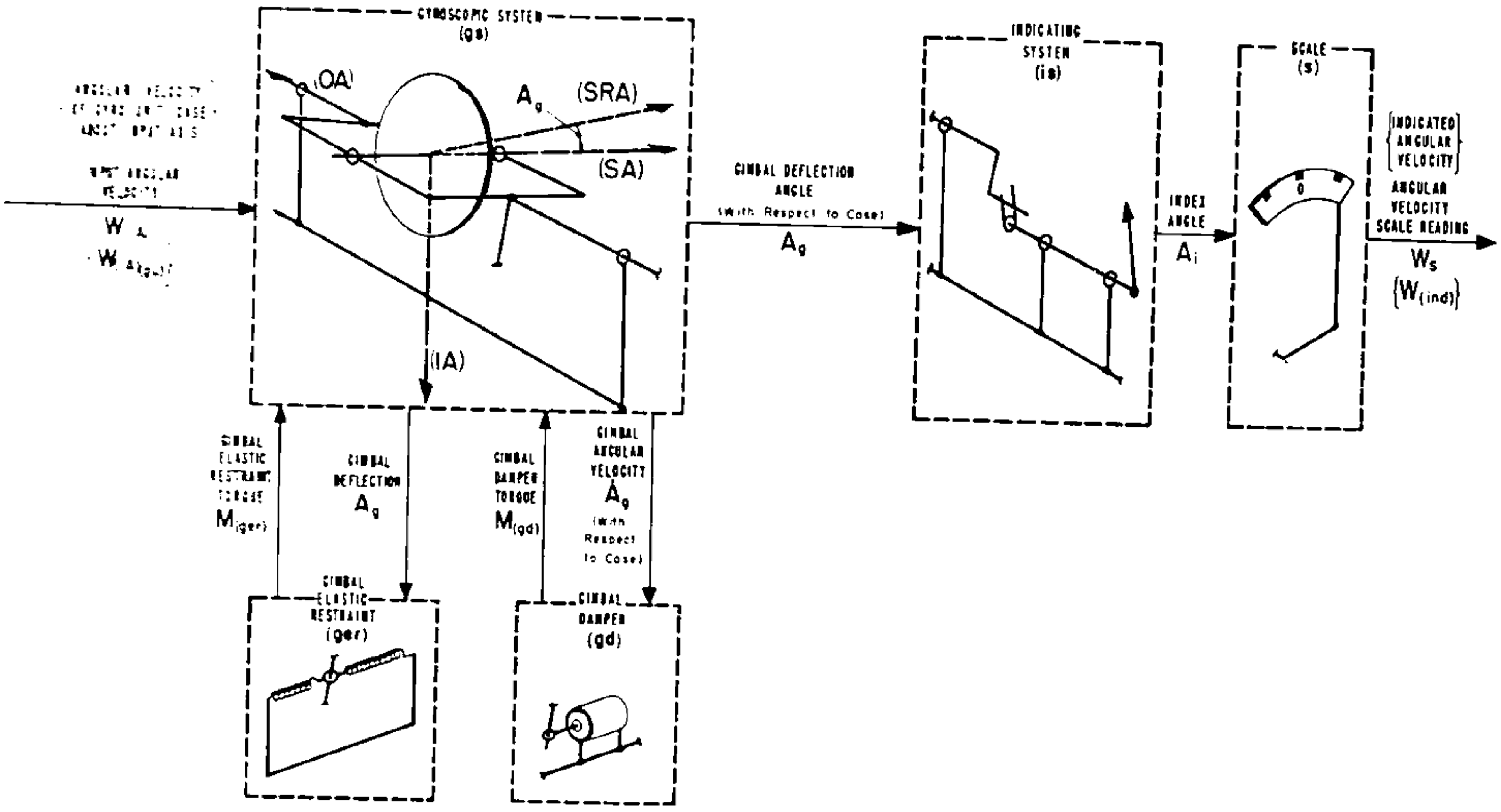


Fig. 7-3. Line schematic-functional diagram for a gyroscopic single-axis angular velocity indicator.

7.7. The output from the indicator is the angular velocity scale reading or the indicated angular velocity. The gimbal elastic restraint receives the gimbal deflection and applies the gimbal elastic restraint torque to the gimbal. The gimbal damper receives the gimbal angular velocity and applies the gimbal damper torque to the gimbal. In operation, rotation of the case about the input axis causes the gyro rotor to apply a torque to the gimbal about the output axis that tends to turn the spin axis into the input axis. This tendency is balanced by the spring under steady-state conditions, and the indicating system causes the index to take up an angle that depends upon the input angular velocity.

FUNCTIONAL DIAGRAM OF A GYROSCOPIC SINGLE-AXIS ANGULAR VELOCITY INDICATOR

7.8. Figure 7-4 is a functional diagram for the instrument of Fig. 7-1. This diagram conveys essentially the same information as the diagram of Fig. 7-3, but is much easier to draw because it contains only rectangular boxes. To facilitate discussions of system operation based on Fig. 7-4, functional names are added to the descriptive names of the pictorial and line schematic diagrams. For example, in Fig. 7-4, the gyro gimbal is called the torque summing member because this member receives applied torque components from the gyroscopic system, the gimbal elastic restraint, and the gimbal damper, and under static conditions balances these components to a zero resultant in the fashion of any rigid body. In Fig. 7-4, heavy full lines are shown connecting the components that are rigidly attached together. An arrowhead on one of these heavy lines shows that the mechanical position* and orientation* of the component at the arrowhead are determined by the component at the originating end of the arrow. In contrast to the rigid connections, medium-weight lines show power-level connections**

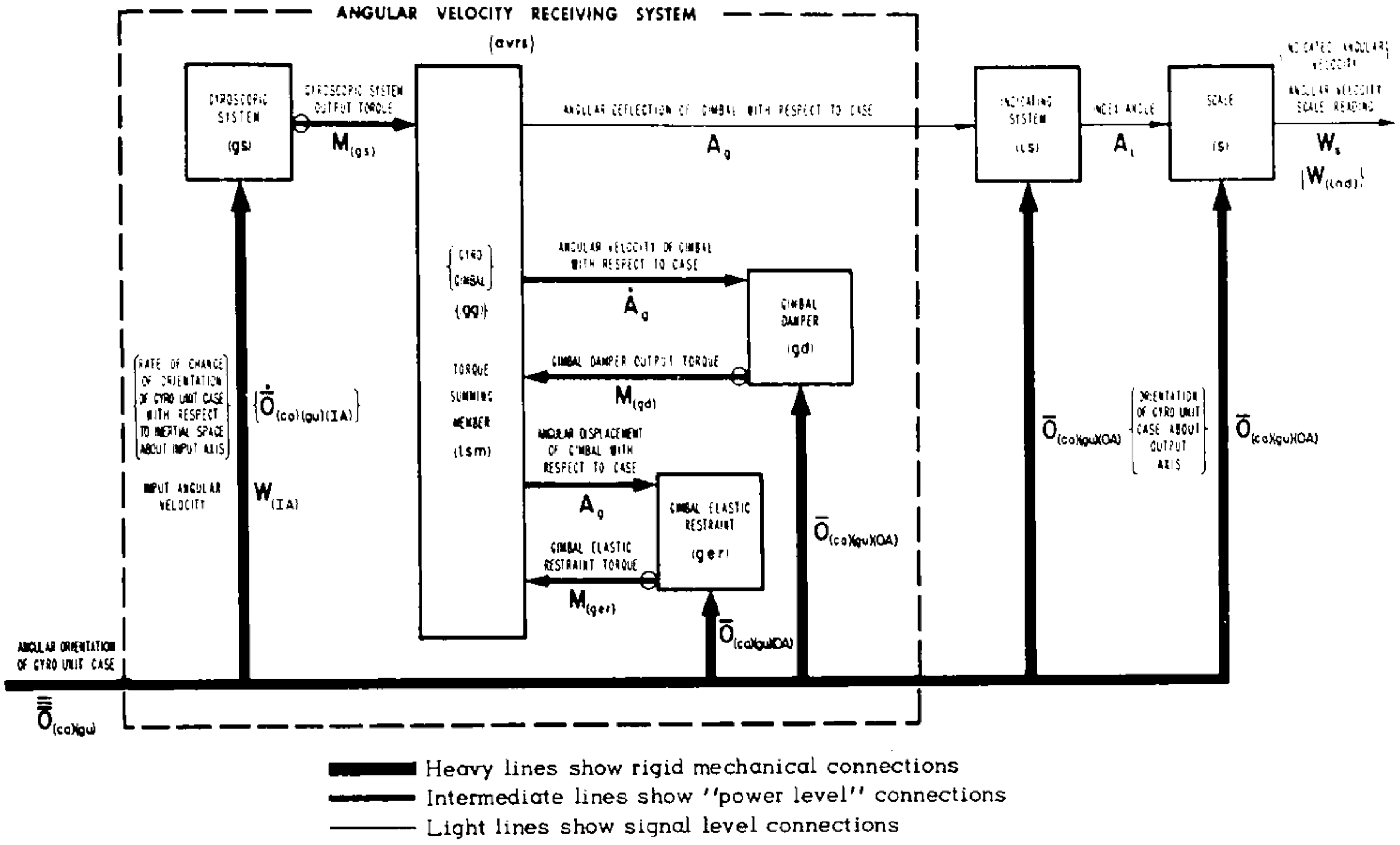

ANGULAR VELOCITY RECEIVING SYSTEM
(avrs)
GYROSCOPIC SYSTEM (gs)
GYROSCOPIC SYSTEM OUTPUT TORQUE
$M_{(gs)}$
ANGULAR DEFLECTION OF GIMBAL WITH RESPECT TO CASE
A_g
{GYRO GIMBAL} (gg)
TORQUE SUMMING MEMBER (tsm)
ANGULAR VELOCITY OF GIMBAL WITH RESPECT TO CASE
$\dot{A}_g$
GIMBAL DAMPER (gd)
GIMBAL DAMPER OUTPUT TORQUE
$M_{(gd)}$
ANGULAR DISPLACEMENT OF GIMBAL WITH RESPECT TO CASE
A_g
GIMBAL ELASTIC RESTRAINT (ger)
GIMBAL ELASTIC RESTRAINT TORQUE
$M_{(ger)}$
$\bar{O}_{(co)(gu)(OA)}$
{RATE OF CHANGE OF ORIENTATION OF GYRO UNIT CASE WITH RESPECT TO INERTIAL SPACE ABOUT INPUT AXIS}
$\{\dot{\bar{O}}_{(co)(gu)(IA)}\}$
INPUT ANGULAR VELOCITY
$W_{(IA)}$
ANGULAR ORIENTATION OF GYRO UNIT CASE
$\bar{\bar{\bar{O}}}_{(co)(gu)}$
INDICATING SYSTEM (is)
INDEX ANGLE
A_i
SCALE (s)
{INDICATED ANGULAR VELOCITY}
ANGULAR VELOCITY SCALE READING
W_s
$\{W_{(ind)}\}$
{ORIENTATION OF GYRO UNIT CASE ABOUT OUTPUT AXIS}
$\bar{O}_{(co)(gu)(OA)}$
Heavy lines show rigid mechanical connections
Intermediate lines show "power level" connections
Light lines show signal level connections

Fig. 7-4. Functional diagram for a gyroscopic single-axis angular velocity indicator.

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* Position implies the location of a point within a body; orientation implies the angular position of a body with respect to established reference coordinates.

** This use of the term power level connection is not strictly in accordance with good practice. Ordinarily, the "power" level in any system is that used for positioning relatively heavy members for some purpose other than the simple transfer of information.

between operating components. Light lines represent signal level connections that act at the lowest power level consistent with the transfer of information. A circle with one of its diameters on a power-level line indicates that the quantity represented is reversibly transferred between the operating components involved. For example, in Fig. 7-4 a circle on the gyroscopic system output torque arrow shows that the torque summing member receives a torque from the gyroscopic system and applies a reaction torque back on this system.

SYMBOLS FOR DESIGNATING ORIENTATION

7.9. In Fig. 7-4, the angular velocity input is shown as being applied to the gyroscopic system through the rigid connection of this system to the case. This rigid connection transfers the orientation of the case to the gyroscopic system. In symbols, orientation is represented by an upper-case letter O. The three short lines, or bars, over the symbol show that three orientational component angles are transferred. For example, the three bars over the symbol for gyro unit case orientation mean that the case has a three-dimensional orientation. The single bar over the respective symbols for the orientation of the gimbal damper, the gimbal elastic restraint, the indicating system and the scale shows that, so far as operation of the system is concerned, the only important orientation for these components is about a single axis. This axis is identified as the output axis of the gyro unit case by the subscript (ca)(gu)(OA); that is, (ca) for case, (gu) for gyro unit, and (OA) for output axis. The orientation symbol for the gyroscopic system carries a single bar to show that only angular motion about a single axis is important. A dot over the bar indicates that the time rate of change of the specified single-axis orientation is the quantity represented. The subscript (ca)(gu)(IA) identifies the axis about which the rotation occurs as the input axis of the gyro unit case. For convenience this rotation is called the input angular velocity and is given the main symbol W, as the nearest English equivalent of the Greek symbol omega, which is conventionally used to represent angular velocity. The subscript (IA) for the main symbol identifies the angular velocity axis as the input axis of the gyro unit.

VECTOR INTERPRETATION OF GYROSCOPIC ACTION

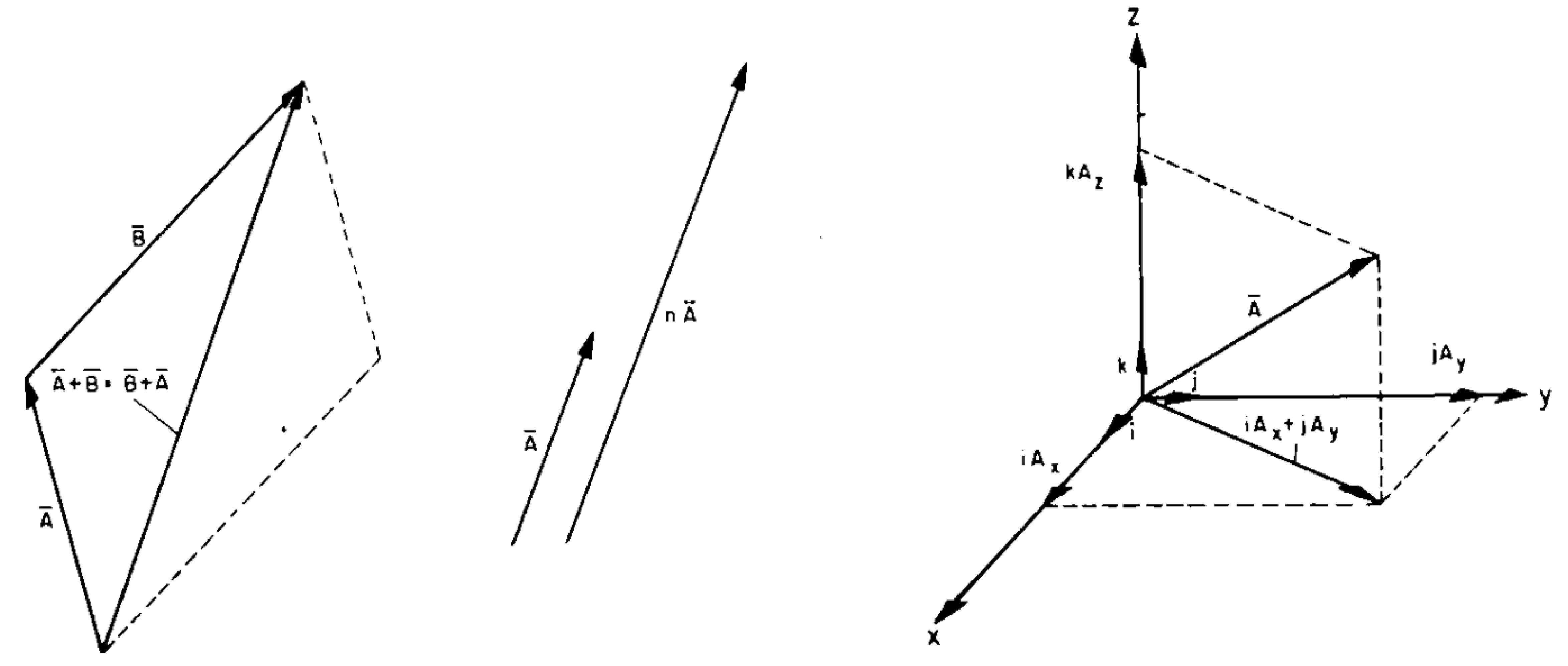
7.10. The action of the gyroscopic system depends on the action of a spinning rotor, which responds to an applied torque as required by Newton's law of dynamics for rotating bodies. For engineering purposes, the important aspects of this dynamical theory are most useful when expressed by means of vectors and developed by the conventions of vector analysis. For this reason, a short review of those elements of vector analysis necessary to an understanding of the present treatment is given here.

VECTOR CONVENTIONS FOR LINEAR QUANTITIES

7.11. Vector analysis offers a general method for dealing with physical quantities, such as forces and linear velocities, which have both magnitude and direction.* In any particular case, a quantity of this nature may be represented by a vector. A vector is a directed segment of a straight line and is usually represented by an arrow. The arrow has a length proportional to the magnitude of the quantity and a sense, or direction, with some convention-defined relationship to the directional property of the quantity. This sense is indicated by the direction of the arrowhead. Physical quantities which have only magnitude instead of both magnitude and direction are called scalars.

ADDITION AND SUBTRACTION OF VECTORS

7.12. Vectors that can be properly drawn from a common reference point may be added according to the parallelogram law illustrated in Fig. 7-5a. To add two given vectors, it is only necessary to place the tail of one vector at the arrowhead of the other vector and draw the resultant vector, which is the sum of the two given vectors, from the tail of the first vector to the head of the second vector. This procedure can be continued to find the sum of any number of vectors in any arbitrary order.



VECTOR SUM $\bar{A}+\bar{B}$ IS DIAGONAL OF PARALLELOGRAM OF WHICH $\bar{A}$ AND $\bar{B}$ ARE ADJACENT SIDES

TO SUBTRACT $\bar{B}$ FROM $\bar{A}$ ADD $-\bar{B}$ TO $\bar{A}$

a) Addition of vectors

n = SCALAR, $\bar{A}$ = VECTOR.
THE PRODUCT $n\bar{A}$:
1) HAS SAME DIRECTION AS $\bar{A}$
2) HAS LENGTH n TIMES THE LENGTH OF $\bar{A}$
3) n HAS ALGEBRAIC SIGN AND SO MAY REVERSE SENSE OF $n\bar{A}$ AS COMPARED TO $\bar{A}$

b) Multiplication of a vector by a scalar

xyz — RIGHT-HAND RECTANGULAR AXIS SYSTEM
i — UNIT VECTOR ALONG + DIRECTION OF x AXIS
j — UNIT VECTOR ALONG + DIRECTION OF y AXIS
k — UNIT VECTOR ALONG + DIRECTION OF z AXIS

IN COMPONENT FORM

$$\bar{A} = iA_x + jA_y + kA_z$$

c) Representation of a vector in terms of rectangular components

Fig. 7-5. Addition, multiplication of vectors by scalars and vector resolution into rectangular components.

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* Only a small part of the branch of mathematics known as vector analysis is required for present purposes. Many treatments of vector analysis have been published; see, for example, Coffin (12) and Phillips (29).

7.13. Vector subtraction is identical with vector addition, except that the direction of the vector to be subtracted is reversed, i.e.,

$$\bar{A} - \bar{B} = \bar{A} + (-\bar{B})$$

MULTIPLICATION OF A VECTOR BY A SCALAR

7.14. Figure 7-5b illustrates the multiplication of a vector by a scalar. Such a product is a vector having the same direction as the original vector, but with its length multiplied by the magnitude of the scalar. The scalar may have an algebraic sign, and so reverse the sense of the original vector, but the direction will always remain unchanged.

REPRESENTATION OF VECTORS IN TERMS OF RECTANGULAR COMPONENTS

7.15. Vectors can be written in terms of their projections on a system of three mutually perpendicular axes. The usual convention is to specify these axes as an x-axis, a y-axis, and a z-axis, where the order of the x, y, and z directions is chosen to form a right-hand system. That is, if a vector that is along the positive direction of the x-axis is turned toward a vector along the positive direction of the y-axis, the rotation will be right-hand about a vector along the positive direction of the z-axis.

7.16. The symbols i, j, and k (in order) are used to represent vectors of unit length along the positive directions of the x-, y-, and z-axes, respectively. Using these unit vectors, any arbitrary vector can be written in terms of its components along a given right-hand system of rectangular axes as

$$\bar{A} = iA_x + jA_y + kA_z$$

where A_x, A_y, and A_z are the magnitudes (with the proper algebraic signs) of the projections of the vector A along the x-, y-, and z-axes, respectively. Figure 7-5c illustrates coordinates for representing a vector in terms of components. The addition or subtraction of vectors expressed in terms of rectangular components can be carried out directly by the addition or subtraction of the corresponding components. Thus,

$$\bar{A} + \bar{B} = i(A_x + B_x) + j(A_y + B_y) + k(A_z + B_z)$$

SCALAR PRODUCT OF TWO VECTORS

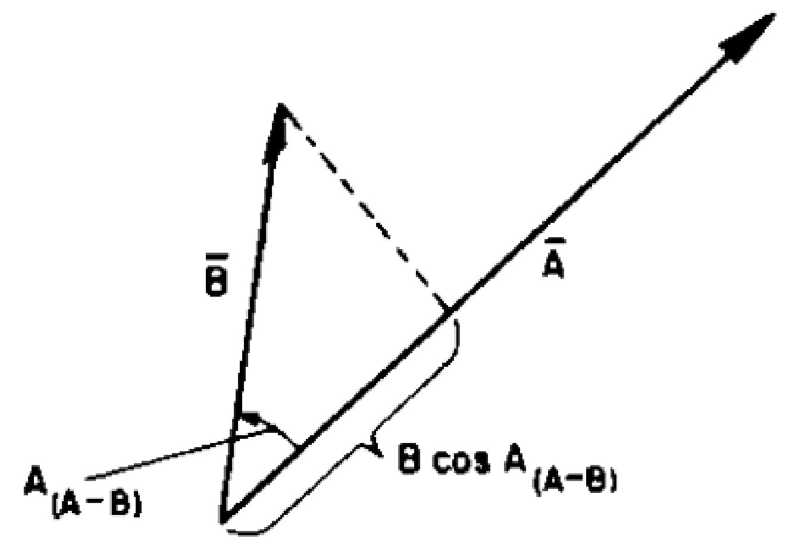
7.17. Two types of vector multiplication are important in elementary vector analysis. In one type, the product of two vectors is a scalar, and for this reason is called the scalar product, or dot product. The result of the other type of multiplication leads to a vector that is called the vector product, or cross product.

7.18. The scalar product of two vectors is the magnitude of one vector multiplied by the magnitude of the projection of the second vector on the first vector. Figure 7-6a illustrates the general nature of the scalar product of two vectors and gives the scalar product relationships in terms of rectangular coordinates.

VECTOR PRODUCT OF TWO VECTORS

7.19. The vector product of two given vectors is a vector perpendicular to the plane determined by the given vectors. Its magnitude is equal to the product of the magnitudes of the given vectors multiplied by the sine of the smaller angle between them. The positive sense of the product vector is determined by the application of the right-hand rule, as illustrated in Fig. 7-6b. This figure also gives the cross product of two vectors in terms of rectangular components. It is important to note that an interchange in the order of the vectors in a cross product reverses the sign of the product.

a) GENERAL DEFINITION



THE SCALAR OR DOT PRODUCT OF TWO VECTORS IS WRITTEN $\bar{A} \cdot \bar{B}$. THIS PRODUCT IS A SCALAR, EQUAL IN MAGNITUDE TO THE MAGNITUDE OF THE FIRST VECTOR MULTIPLIED BY THE MAGNITUDE OF THE PROJECTION OF THE SECOND VECTOR ON THE FIRST VECTOR.

$$\bar{A} \cdot \bar{B} = \bar{B} \cdot \bar{A} = AB \cos A_{(A-B)}$$

WHERE $A_{(A-B)}$ IS THE ANGLE BETWEEN $\bar{A}$ AND $\bar{B}$ MEASURED FROM $\bar{A}$ TO $\bar{B}$

b) THE SCALAR PRODUCT IN TERMS OF RECTANGULAR COORDINATES.

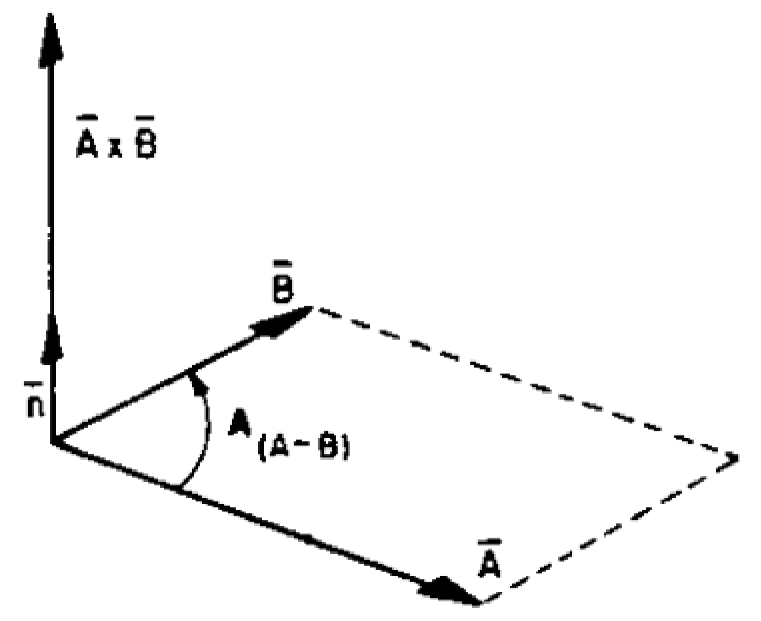
DOT PRODUCTS OF UNIT VECTORS

$$i \cdot i = j \cdot j = k \cdot k = 1$$
$$i \cdot j = i \cdot k = j \cdot k = 0$$

$$\bar{A} \cdot \bar{B} = (iA_x + jA_y + kA_z) \cdot (iB_x + jB_y + kB_z)$$
$$= A_x B_x + A_y B_y + A_z B_z$$

a) Scalar or dot product of two vectors

a) GENERAL DEFINITION



THE VECTOR PRODUCT OF TWO VECTORS IS WRITTEN $\bar{A} \times \bar{B}$ AND IS BY DEFINITION A VECTOR PERPENDICULAR TO THE PLANE OF $\bar{A}$ AND $\bar{B}$. THE POSITIVE SENSE OF THE VECTOR $\bar{A} \times \bar{B}$ IS RELATED TO THE VECTORS $\bar{A}$ AND $\bar{B}$ BY THE RIGHT-HAND RULE; i.e., THE POSITIVE SENSE OF $\bar{A} \times \bar{B}$ IS GIVEN BY ROTATING $\bar{A}$ (THE FIRST TERM IN THE PRODUCT) TOWARD $\bar{B}$ THROUGH THE SMALLER ANGLE. THIS MEANS THAT THE ORDER OF $\bar{A}$ AND $\bar{B}$ IN THE PRODUCT IS IMPORTANT. A REVERSAL IN ORDER MEANS A REVERSAL IN THE SIGN OF THE PRODUCT; i e.,

$$\bar{A} \times \bar{B} = -\bar{B} \times \bar{A}$$

THE MAGNITUDE OF THE CROSS PRODUCT IS THE PRODUCT OF THE MAGNITUDES OF THE TWO VECTORS MULTIPLIED BY THE SINE OF THE SMALLER ANGLE BETWEEN THEM, i.e., $\bar{A} \times \bar{B} = \bar{n}\, AB \sin A_{(A-B)}$
WHERE $\bar{n}$ IS A UNIT VECTOR NORMAL TO THE PLANE CONTAINING $\bar{A}$ AND $\bar{B}$

b) THE VECTOR PRODUCT IN TERMS OF RECTANGULAR COORDINATES.

CROSS PRODUCTS OF UNIT VECTORS

$$i \times i = j \times j = k \times k = 0$$
$$i \times j = k, \; j \times k = i, \; k \times i = j$$
$$j \times i = -k, \; k \times j = -i, \; i \times k = -j$$

$$\bar{A} \times \bar{B} = (iA_x + jA_y + kA_z) \times (iB_x + jB_y + kB_z)$$
$$= i(A_y B_z - A_z B_y) + j(A_z B_x - A_x B_z) + k(A_x B_y - A_y B_x) = \begin{vmatrix} i & j & k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

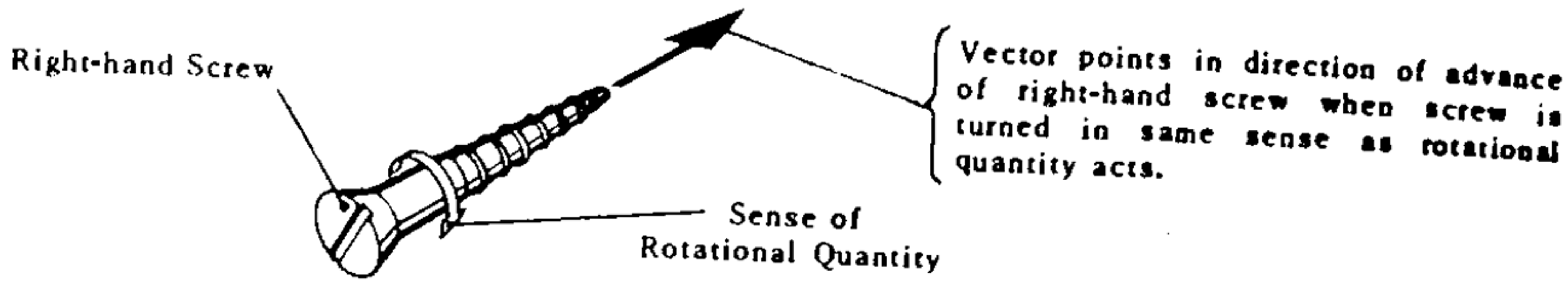
b) Vector or cross product of two vectors

Fig. 7-6. Products of two vectors.

7.20. The triple cross product of three vectors, in the form $\bar{A} \times (\bar{B} \times \bar{C})$, can be resolved into the following expression that is frequently useful in operating with vector quantities:

$$\bar{A} \times (\bar{B} \times \bar{C}) = \bar{B}(\bar{A} \cdot \bar{C}) - \bar{C}(\bar{A} \cdot \bar{B})$$

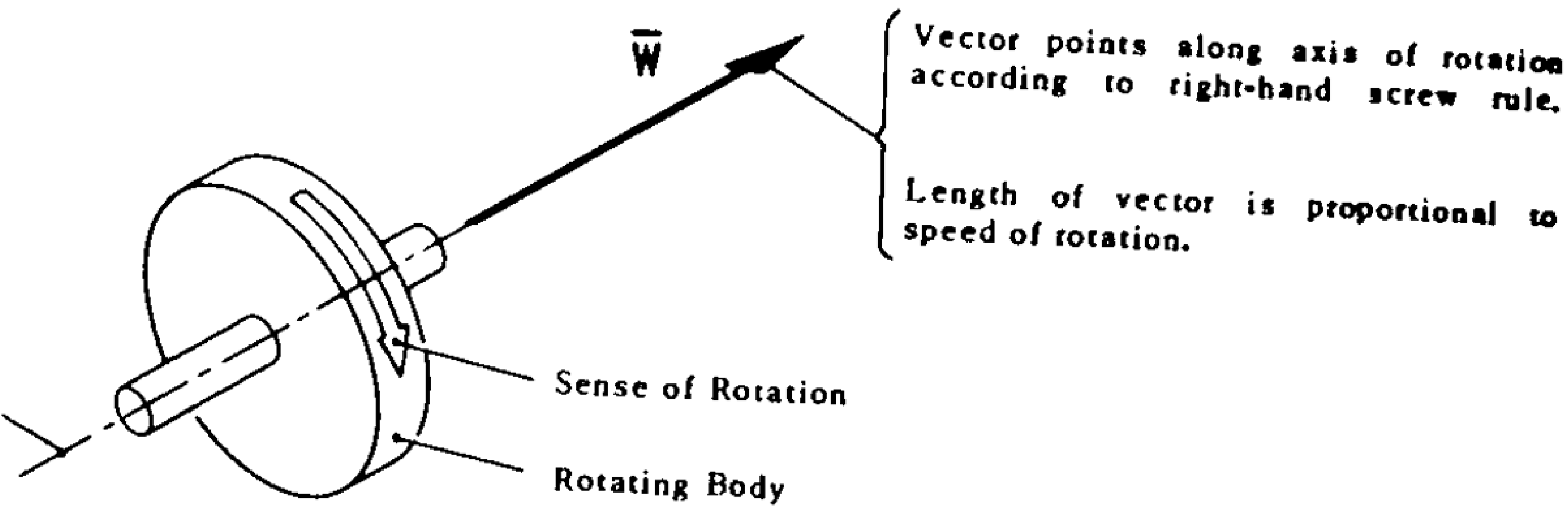
7.21. Figure 7-7 gives a summary of the definitions and conventions that are of primary importance for the theory of gyroscopic action. Figure 7-8a is a line schematic diagram showing the essential features of the gyroscopic element,


Right-hand Screw
Vector points in direction of advance of right-hand screw when screw is turned in same sense as rotational quantity acts.
Sense of Rotational Quantity

a) Right-hand screw rule for the relationship between a rotational quantity and the vector representing this quantity.

$\bar{W}$ = Symbol for an angular velocity vector

W = Symbol for speed of rotation


Axis of Rotation
W̄
Vector points along axis of rotation according to right-hand screw rule.
Length of vector is proportional to speed of rotation.
Sense of Rotation
Rotating Body

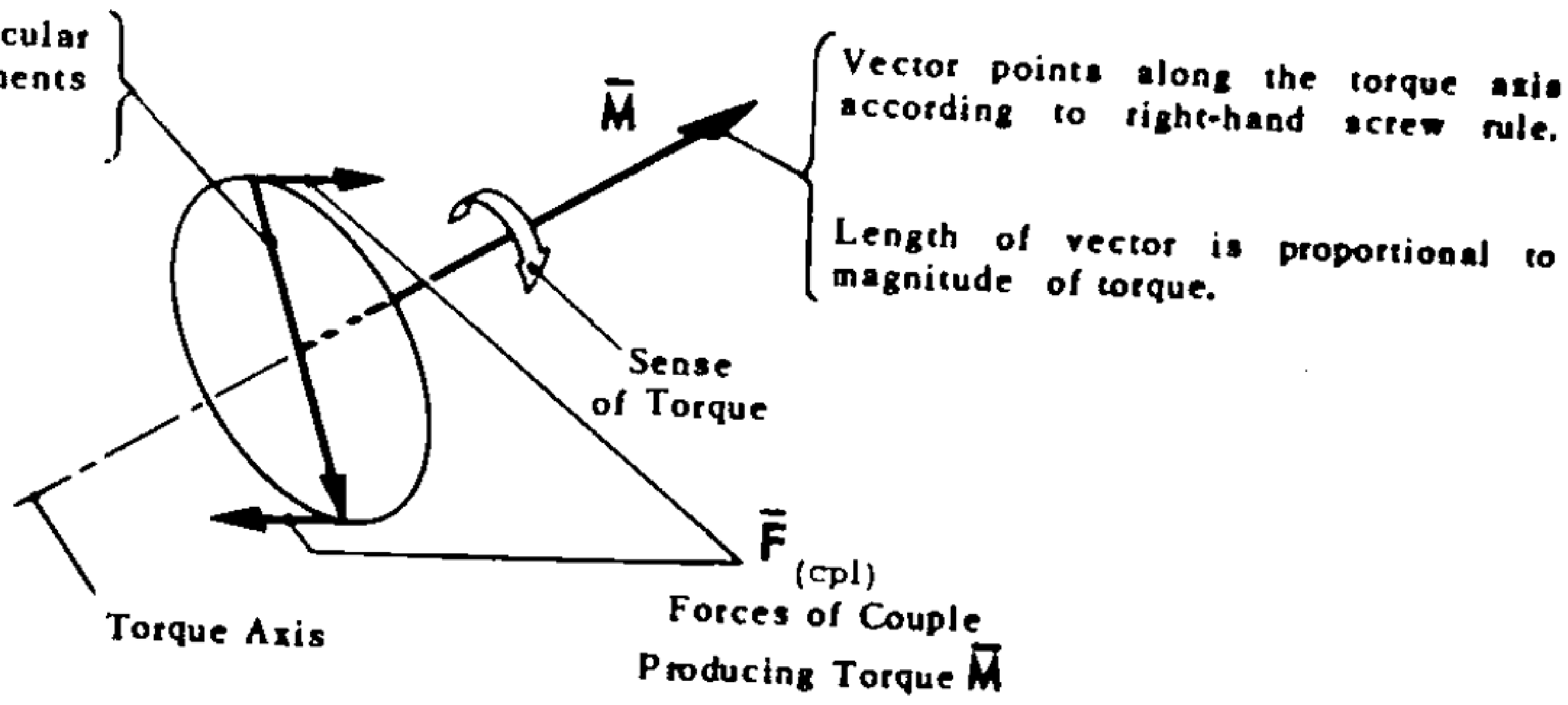
b) Vector representation of angular velocity.

$\bar{\ell}$ = Vector representing perpendicular arm between force components producing torque

ℓ = Magnitude of $\bar{\ell}$

$\bar{M}$ = Vector Symbol for a torque

M = Magnitude of the torque = $F_{(cpl)}\ell$


M̄
Vector points along the torque axis according to right-hand screw rule.
Length of vector is proportional to magnitude of torque.
Sense of Torque
Torque Axis
F̄ (cpl)
Forces of Couple Producing Torque M̄

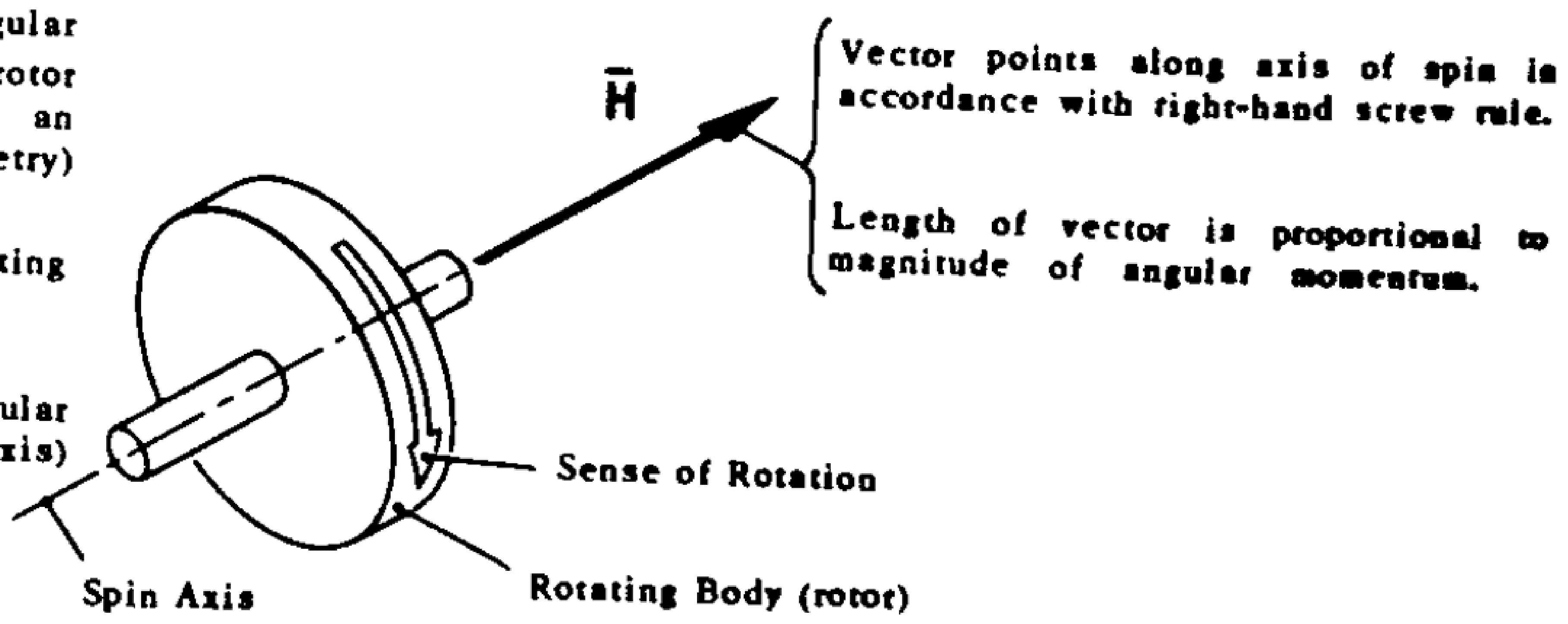
For this diagram the term "torque" has a meaning identical with the word "couple" as used in conventional mechanics.

c) Vector representation of a torque.

$\bar{H} = I_{(sp)}\bar{W}_{(sp)}$ = Vector angular momentum of rotor (spinning about an axis of symmetry)

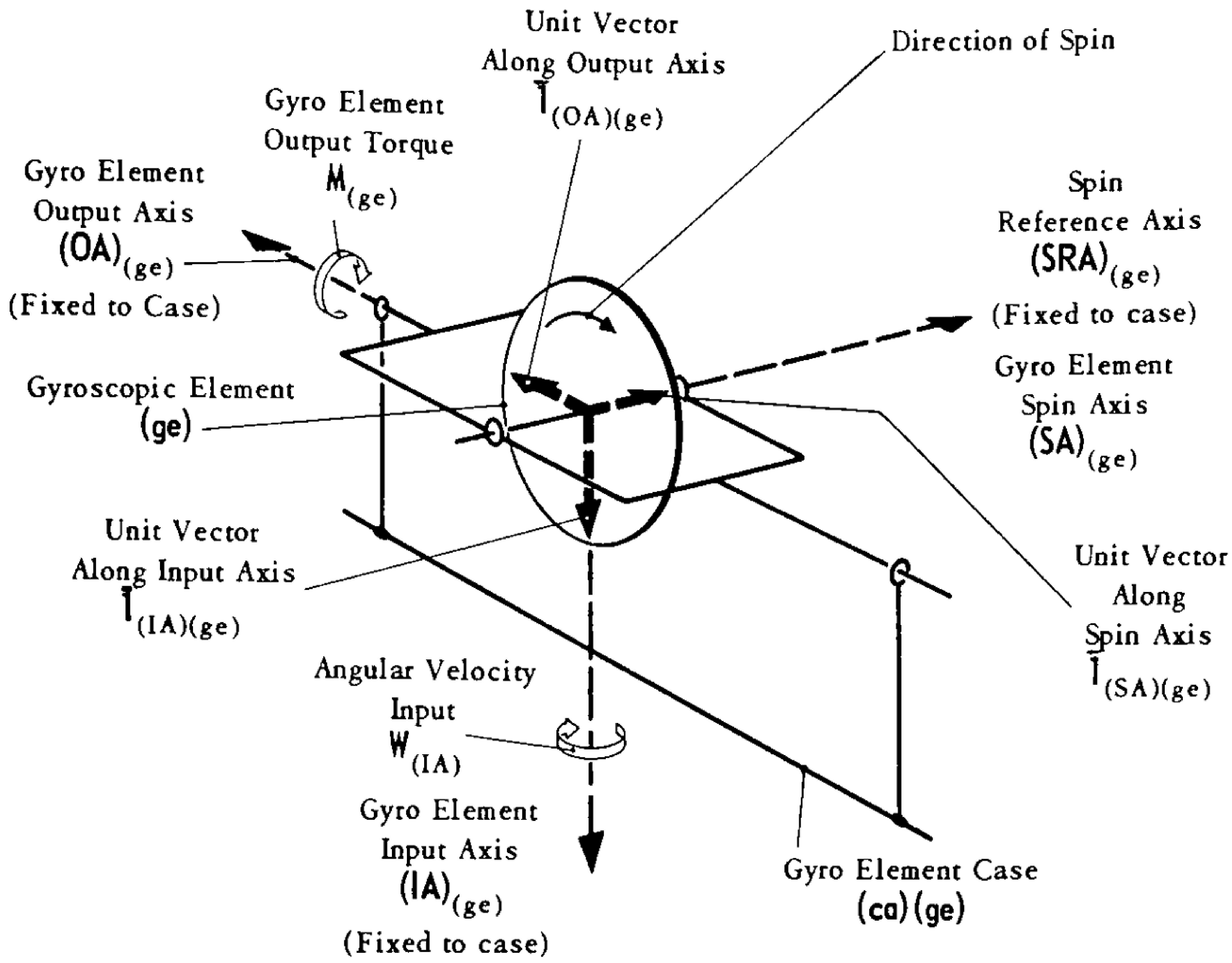
$I_{(sp)}$ = Moment of inertia of rotating body about spin axis

$\bar{W}_{(sp)}$ = Spin velocity (vector angular velocity about spin axis)

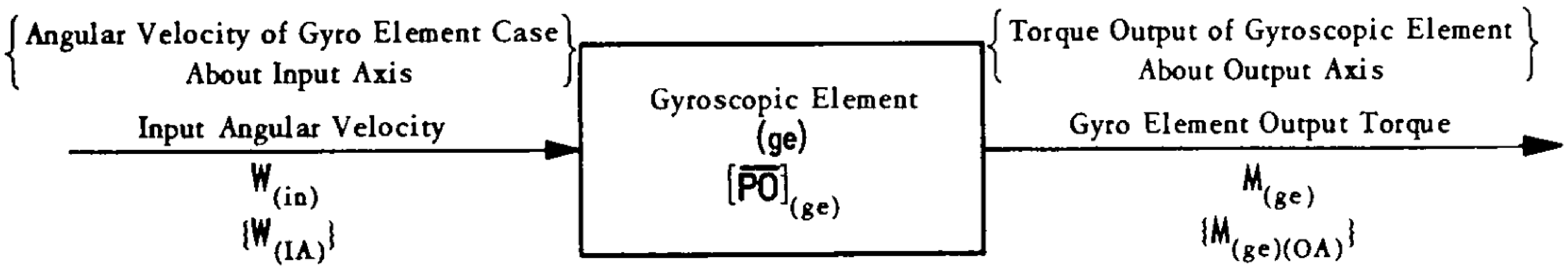

H̄
Vector points along axis of spin in accordance with right-hand screw rule.
Length of vector is proportional to magnitude of angular momentum.
Sense of Rotation
Spin Axis
Rotating Body (rotor)

d) Vector representation for the angular momentum of a spinning rotor.

Fig. 7-7. Vector representation of rotational quantities.



a) Line schematic diagram of a single-axis gyroscopic element


Angular Velocity of Gyro Element Case About Input Axis
Input Angular Velocity
$W_{(in)}$
$\{W_{(IA)}\}$
Gyroscopic Element
(ge)
$[\overline{PO}]_{(ge)}$
Torque Output of Gyroscopic Element About Output Axis
Gyro Element Output Torque
$M_{(ge)}$
$\{M_{(ge)(OA)}\}$

$[\overline{PO}]_{(ge)} \equiv$ Gyroscopic element performance operator

$$[\overline{PO}]_{(ge)} \equiv H[\bar{1}_{(SRA)(ge)} \times (\;)]^{①②}$$

$$\bar{M}_{(ge)} = H[\overline{PO}]_{(ge)} \times \bar{W}_{(IA)}$$

$$= H(\bar{1}_{(SRA)(ge)} \times \bar{W}_{(IA)})$$

$$= H(\bar{1}_{(SRA)(ge)} \times \bar{1}_{(IA)(ge)})W_{(IA)(ge)}$$

$$M_{(ge)} = HW_{(IA)(ge)}$$

1. See Derivation Summary 7-1 for derivation of this operator.

2. This form for the operator depends on the assumption that the angle between $(SRA)_{(ge)}$ and $(SA)_{(ge)}$ is negligibly small.

By assumption $\bar{H} = \bar{1}_{(SRA)(ge)}H$; Magnitude of $\bar{H} \doteq$ CONSTANT, i.e., $= H =$ CONSTANT

By vector conventions of Fig. 7-5

$$\bar{W}_{(IA)} = \bar{1}_{(IA)(ge)}W_{(IA)}; \quad \bar{M}_{(ge)(OA)} = \bar{1}_{(OA)(ge)}M_{(ge)(OA)}$$

b) Functional diagram of a single-axis gyroscopic element

Fig. 7-8. Performance characteristics of a single-axis gyroscopic element.

which is a mathematical entity representing the dynamic action of a gyroscopic rotor in terms of a simple angular momentum vector, without recognizing the properties that are not associated with the spinning motion. Figure 7-8b is a functional diagram representing the essential performance characteristics of the gyroscopic element. The essential action of a gyroscopic rotor spinning about its axis of symmetry is to receive an angular velocity input about an axis normal to the spin axis and to generate a torque about an output axis that is mutually perpendicular to both the input axis and the spin axis. This three-dimensional space situation is not describable in terms of scalar quantities, but requires the use of vector concepts that may be formally represented by a vector performance operator. As shown in Fig. 7-8b, the basic symbol for this operator is formed by the capital letters PO with an overline, or superscript bar, and enclosed in square brackets. The operator symbol is identified with the component whose behavior it describes by means of subscripts. In Fig. 7-8b, this subscript is (ge) to show that the operator is associated with the gyroscopic element.

In general, by Newton's Dynamical Law for Rotation

$$\left(\frac{d\bar{H}}{dt}\right)_I = \bar{M}_{(app)}$$

$\left(\frac{d\bar{H}}{dt}\right)_I$ = Time rate of change of angular momentum with respect to inertial space ; $\bar{M}_{(app)}$ = Torque applied to the gyroscopic element

Because $\bar{H}$ is constant in magnitude the time rate of change of $\bar{H}$ must be at right angles to $\bar{H}$ itself. That is, the $\bar{H}$ vector will rotate with respect to inertial space with an angular velocity $\bar{W}_{IH}$ given by the expression

$$\left(\frac{d\bar{H}}{dt}\right)_I = \bar{M}_{(app)} = \bar{W}_{IH} \times \bar{H} ①$$

If the torque applied to the gyroscopic element, $\bar{M}_{(app)}$ is replaced by the torque applied by the gyroscopic element to external restraints; i.e., $\bar{M}_{(ge)(out)} = \bar{M}_{(app)}$

$$\bar{M}_{(ge)(out)} = -\bar{W}_{IH} \times \bar{H} = \bar{H} \times \bar{W}_{IH}$$

For the special case of the single-degree-of-freedom gyroscopic element given a forced angular velocity of the case about the input axis $(IA)_{(ge)}$ while the spin axis is aligned with the spin reference axis

$$\bar{M}_{(ge)(out)} = H[\bar{1}_{(SRA)(ge)} \times \bar{W}_{(IA)(ge)}]$$

$$= H[\bar{1}_{(SRA)(ge)} \times (\bar{\ })] ② \bar{W}_{(IA)(ge)} ③$$

so that

$$\overline{[PO]}_{(ge)} = H[\bar{1}_{(SRA)(ge)} \times (\bar{\ })]$$

1. The background of vector analysis for this expression is given in Chapter 9.
2. Square brackets, [ ], show that the enclosed symbol represents not a simple factor of a product but is a performance operator. A bar over the bracket indicates that the operator has vector properties. The symbol $\overline{[PO]}$ represents the generalized performance operator.
3. The open parentheses inside the performance operator brackets indicates that an input quantity with vector properties is to be put in the place of the parentheses.

Derivation Summary 7-1. Derivation of vector performance operator for a single-axis gyroscopic element.

DERIVATION SUMMARY FOR A SINGLE-DEGREE-OF-FREEDOM GYROSCOPIC ELEMENT

7.22. Derivation Summary 7-1 illustrates a generally useful method for outlining the theory behind the performance operator associated with a given operating component. A derivation summary is not intended to present a complete treatise on the theory of an operating component, but should give enough information to recall the details of a once-familiar analysis or to reduce the time and effort required for a qualified engineer using standard references to understand the theoretical background for the behavior of a given device.

7.23. Derivation Summary 7-1 shows Newton's dynamical law for rotation expressed in vector terms. The basic principle is that the time rate of change of angular momentum with respect to inertial space is equal to the applied torque. When the body with which the angular momentum is associated is symmetrical about an axis, when it rotates about this axis of symmetry with a relatively high angular velocity of spin, and when the body has only very small angular velocity components about axes at right angles to the spin axis, the essential dynamical action may be represented in terms of a single vector. When this situation exists, the rotating body has the performance characteristics of the gyroscopic element. In order for a spinning rotor to be describable as a gyroscopic element, it must have its spin angular velocity maintained at a constant magnitude. This means that the vector representing the gyroscopic element has a constant length. As a consequence, a vector of this kind can change with respect to time only by changing its direction.

GYROSCOPIC ANGULAR MOMENTUM

7.24. In Derivation Summary 7-1, the time rate of change of the gyroscopic element angular momentum vector must be associated with a change of direction that in physical terms is simply an angular velocity of the vector. When the applied torque is represented by a vector under the convention of Fig. 7-7c, and the angular velocity of the angular momentum vector is represented under the convention of Fig. 7-7b, Newton's law for rotation is expressed by the equality of the applied torque vector to the cross product of the angular velocity of the angular momentum vector and the angular momentum vector itself. This representation shows that when the angular momentum magnitude is constant, the applied torque vector, the angular momentum vector, and the angular velocity of the angular momentum vector are mutually at right angles to each other. The physical situation described by the gyroscopic element performance equation is that the angular momentum vector rotates toward the applied torque vector. In effect, the angular momentum component turns so that it increases its projection on the applied torque vector at a rate that is proportional to the magnitude of the applied torque.

7.25. Engineering applications of gyroscopic theory are often conveniently discussed in terms of the torque applied by a spinning rotor to restraints that act on the mechanical system supporting the rotor. In Derivation Summary 7-1, the torque applied by the gyroscopic element to its restraints has a magnitude equal to the magnitude of the applied torque, while its direction is opposite to that of the applied torque. Because this torque applied by the gyroscopic element to its restraints has the nature of an output from the element, it is useful to identify the torque as the gyroscopic element output torque. In Derivation Summary 7-1, this quantity is given the main symbol M, with a superscript bar to show its vector nature and with the subscripts (ge) for gyroscopic element and (out) to identify the torque as the output from the gyroscopic element. The mathematical equation for the gyroscopic element output torque is identical with that for the applied torque, except that the reversed sign is obtained by interchanging the angular velocity and angular momentum symbols.

VECTOR PERFORMANCE OPERATOR FOR THE GYROSCOPIC ELEMENT

7.26. The vector performance operator for the gyroscopic element is formed by placing the unit vector for the angular momentum vector and an open pair of parentheses with a superscript bar within the performance operator bracket. This form is shown in Derivation Summary 7-1, which also indicates how the operator relates the output torque to an angular velocity input.

SCALAR PERFORMANCE EQUATION FOR THE GYROSCOPIC ELEMENT

7.27. The development of Derivation Summary 7-1 justifies the performance operator forms of Fig. 7-8b. The diagram of Fig. 7-8a defines the concepts required to describe the behavior of a single-degree-of-freedom gyroscopic element. In this figure, the angular momentum vector is assumed to be along the spin axis, which is at right angles to the output axis. The input axis is fixed to the case at right angles to the output axis. The spin reference axis is fixed to the case at right angles to both the output axis and the input axis. When the case of the single-degree-of-freedom gyroscopic element is forced to have an angular velocity about the input axis, the angular momentum vector tends to turn toward alignment with the angular velocity input axis. The torque corresponding to this tendency is the output torque from the gyroscopic element. The functional diagram of Fig. 7-8b shows that the actuating input for the gyroscopic element is the angular velocity of the case about the input axis and that the corresponding output is the torque from the element acting about the output axis. The performance operator given in Fig. 7-8b describes the situation in which the spin axis is coincident with the spin reference axis and shows that the magnitude of the output torque is equal to the magnitude of the input axis angular velocity multiplied by the magnitude of the gyroscopic element angular momentum. This scalar performance equation carries with it algebraic signs for all the quantities involved. These signs are all associated with the definitions and rotational senses of Fig. 7-8 by the vector conventions of Fig. 7-7.

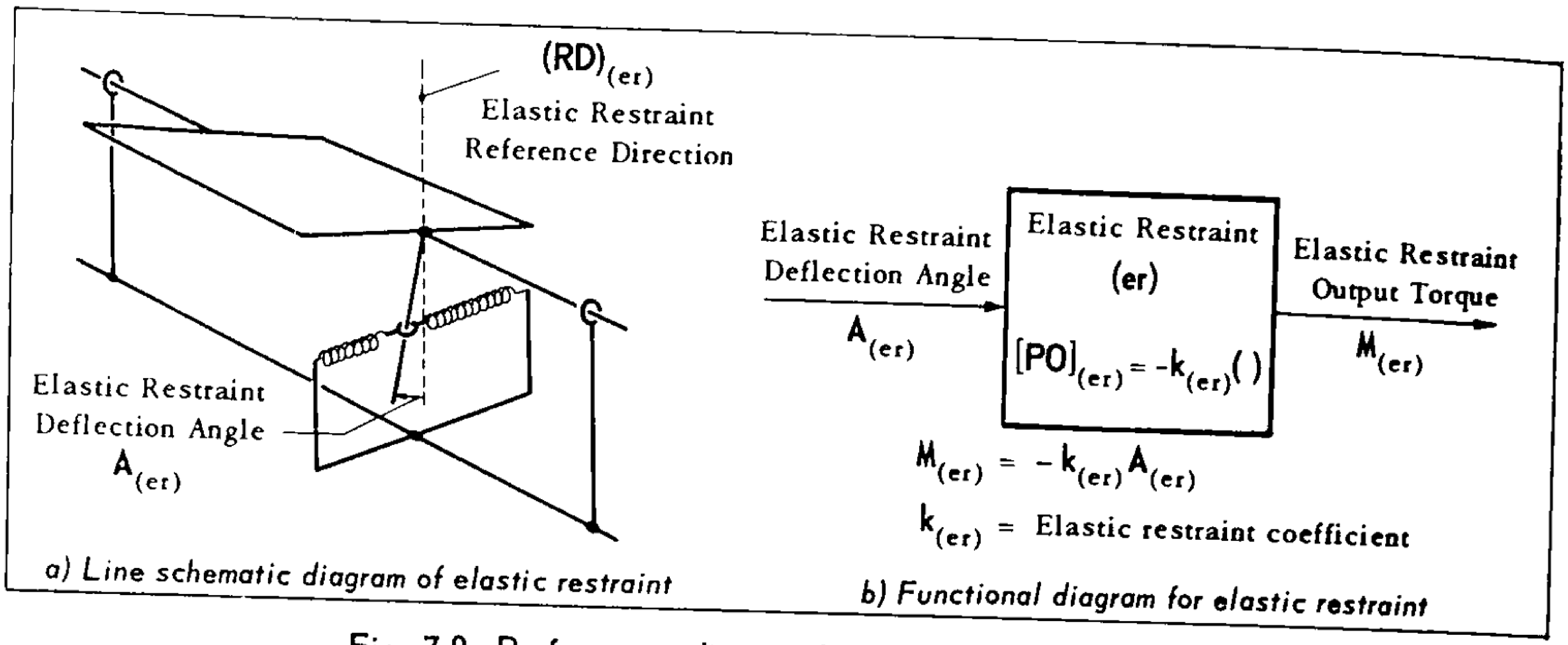
PERFORMANCE CHARACTERISTICS FOR GIMBAL ELASTIC RESTRAINT, GIMBAL DAMPER, AND TORQUE SUMMING MEMBER

7.28. Figures 7-9, 7-10, and 7-11 summarize performance information on the gimbal elastic restraint, the gimbal damper, and the torque summing member respectively, of the gyroscopic single-degree-of-freedom angular velocity indicator of Fig. 7-1. The components represented by these figures are all so simple that the diagrams given are self-explanatory and require no special comment here. When this desirable situation exists all the necessary information on the performance of the components that form an operating system may be read directly from simple diagrams.

THE PERFORMANCE EQUATION DIAGRAM

7.29. Figure 7-12 illustrates the performance equation diagram for summarizing the essential performance information associated with an operating system. Individual operating components are represented by rectangular boxes labeled only by the identifying symbols for the components. Inputs and outputs are shown by arrows, with the outputs related to the corresponding inputs by equations taken from the performance diagrams for individual operating components. In many situations, it is useful to combine a group of operating components into a subsystem which acts as a simple link in the over-all system chain. In Fig. 7-12 the gyroscopic element, the torque summing member, the gimbal damper, and the gimbal elastic restraint are grouped together by a dashed-line box, which is identified as the angular velocity receiving system. The performance equation for this system is formed by combining the performance equations for the individual operating components, and is placed inside the dashed-line box.

7.30. In Fig. 7-12 the angular velocity receiving system receives the angular velocity of the indicator case about its input axis and generates the gimbal deflection angle. The performance equation for the system shows that the input-output relationship involved is not a simple ratio, but depends upon the deflection angle of the torque summing member (the gyro gimbal) and the first and second time derivatives of this angle. The gimbal deflection angle is received by the indicating system, which produces the index angle as its output. The output from the complete indicator is the indicated angular velocity, which is the angular velocity reading that corresponds to the index position with respect to the scale.



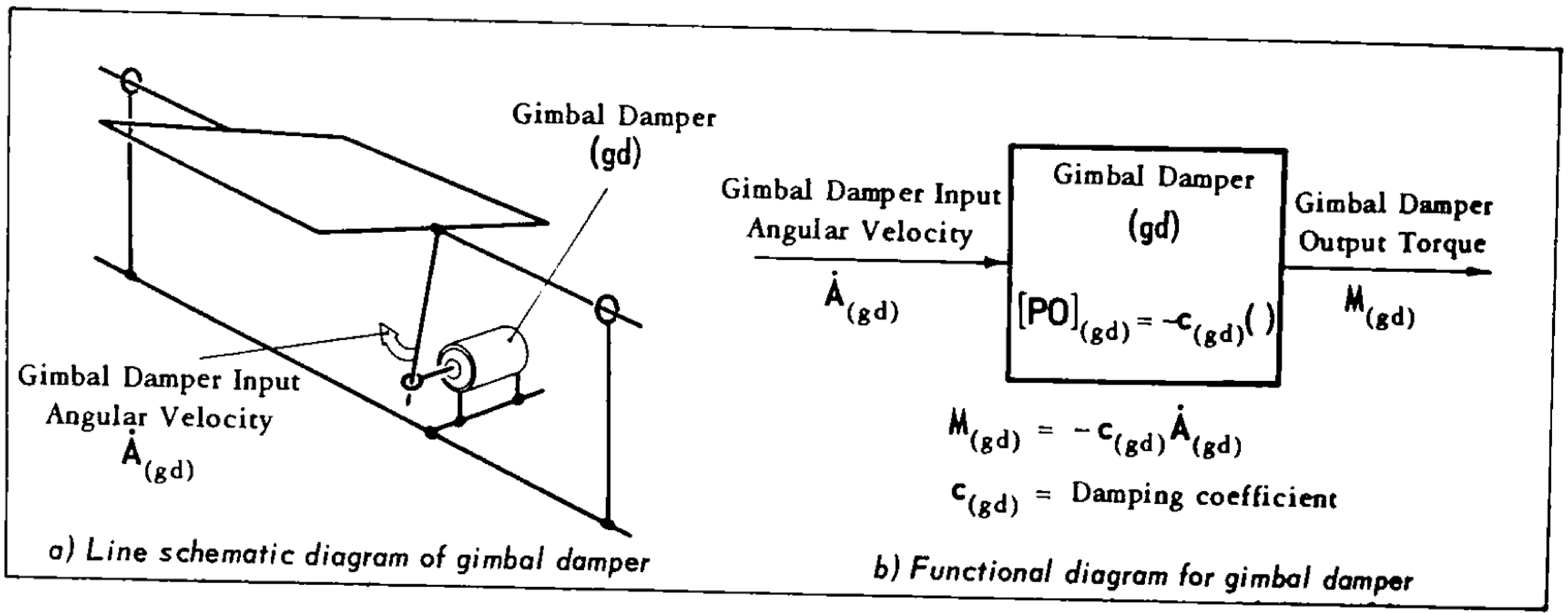
$$M_{(er)} = -k_{(er)} A_{(er)}$$

$k_{(er)}$ = Elastic restraint coefficient

a) Line schematic diagram of elastic restraint

b) Functional diagram for elastic restraint

Fig. 7-9. Performance diagram for gimbal elastic restraint.



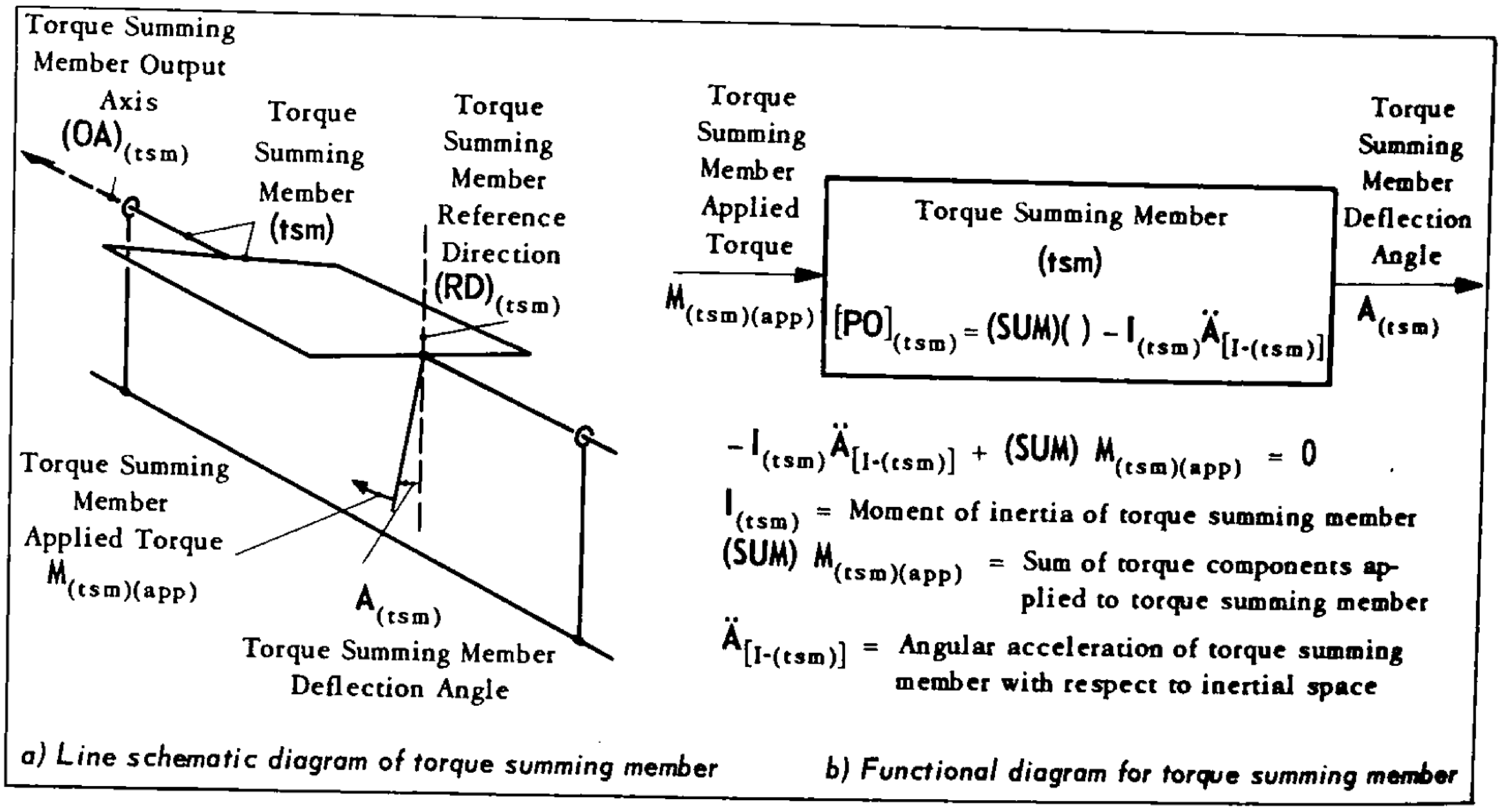
$$M_{(gd)} = -c_{(gd)} \dot{A}_{(gd)}$$

$c_{(gd)}$ = Damping coefficient

a) Line schematic diagram of gimbal damper

b) Functional diagram for gimbal damper

Fig. 7-10. Performance diagram for gimbal damper.



$$-I_{(tsm)} \ddot{A}_{[I\text{-}(tsm)]} + (SUM)\, M_{(tsm)(app)} = 0$$

$I_{(tsm)}$ = Moment of inertia of torque summing member

$(SUM)\, M_{(tsm)(app)}$ = Sum of torque components applied to torque summing member

$\ddot{A}_{[I\text{-}(tsm)]}$ = Angular acceleration of torque summing member with respect to inertial space

a) Line schematic diagram of torque summing member

b) Functional diagram for torque summing member

Fig. 7-11. Performance diagram for torque summing member.

7.31. The over-all performance equation expressed solely in terms of the input and output quantities of the complete indicator is formed from the angular velocity receiving system performance function modified by the indicating system and scale sensitivities.

7.32. The performance equation derivation for the angular velocity indicator illustrates one way of summarizing in quantitative form the essential information for an operating system. No hard and fast rules exist for presentations of this kind, so that in any given case the essential data may be given by diagrams differing considerably from those discussed above, and the performance may be described by other means than a performance equation like that in Fig. 7-12. However for engineering purposes, whatever the method of data presentation used in any particular case, the information represented should include at least all the facts summarized in the series of diagrams and the differential equation described for the gyroscopic angular velocity indicator.

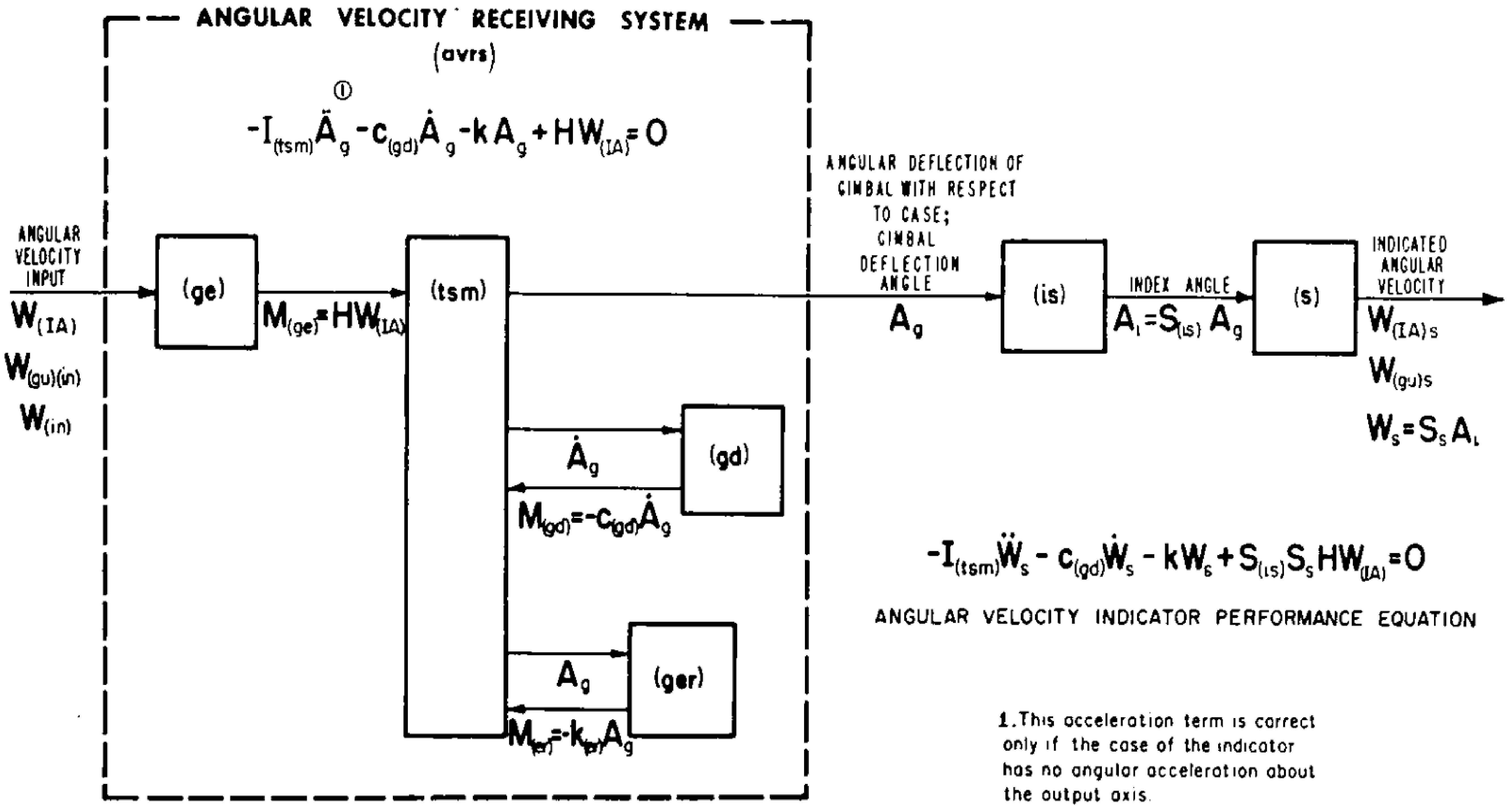

ANGULAR VELOCITY RECEIVING SYSTEM
(avrs)
①
$-I_{(tsm)}\ddot{A}_g - c_{(gd)}\dot{A}_g - kA_g + HW_{(IA)} = 0$
ANGULAR VELOCITY INPUT
$W_{(IA)}$
$W_{(gu)(in)}$
$W_{(in)}$
(ge)
$M_{(ge)} = HW_{(IA)}$
(tsm)
$\dot{A}_g$
(gd)
$M_{(gd)} = -c_{(gd)}\dot{A}_g$
A_g
(ger)
$M_{(er)} = -k_{(er)}A_g$
ANGULAR DEFLECTION OF GIMBAL WITH RESPECT TO CASE; GIMBAL DEFLECTION ANGLE
A_g
(is)
INDEX ANGLE
$A_i = S_{(is)} A_g$
(s)
INDICATED ANGULAR VELOCITY
$W_{(IA)s}$
$W_{(gu)s}$
$W_s = S_s A_i$
$-I_{(tsm)}\ddot{W}_s - c_{(gd)}\dot{W}_s - kW_s + S_{(is)}S_s HW_{(IA)} = 0$
ANGULAR VELOCITY INDICATOR PERFORMANCE EQUATION
1. This acceleration term is correct only if the case of the indicator has no angular acceleration about the output axis.

Fig. 7-12. Performance equation diagram for a gyroscopic single-axis angular velocity indicator.

GENERALIZED PERFORMANCE EQUATION FORM

7.33. Performance equations for operating systems of all kinds may be made to fall into a pattern of which that for the gyroscopic single-axis angular velocity indicator is typical. The basic mathematical form for these equations expresses the fact that a sum of terms involving an output quantity is equal to a sum of terms involving some input quantity. Each of the output terms involves the first power of a derivative of some order of the output with respect to time. With the possible exception of a constant term, each input contains the first power of a derivative of some order of the input with respect to time. In general, coefficients appear as positive multipliers for the derivative terms. The complete group of these coefficients contains the essential information on the performance of the system associated with the equation. The basic features of a typical performance equation are illustrated in the form

$$a_n \frac{d^n q_{out}}{dt^n} + a_{n-1} \frac{d^{n-1} q_{out}}{dt^{n-1}} + \dots + a_1 \frac{dq_{out}}{dt} + a_o q_{out}$$

$$= b_{cnst} + b_o q_{in} + b_1 \frac{dq_{in}}{dt} + \dots + b_{m-1} \frac{d^{m-1} q_{in}}{dt^{m-1}} + b_m \frac{d^m q_{in}}{dt^m} \qquad (7\text{-}1)$$

In any particular case, the terms of Eq. (7-1) may represent approximations to the actual physical facts. For example, a mechanical spring like that described as the elastic restraint for the gyroscopic angular velocity indicator gimbal might not be linear; i.e., the output torque might not be proportional to the angular displacement input. This fact could be recognized by using an *effective coefficient* to describe the spring action. The loss in accuracy due to properly made approximations of this type is usually less serious than the increase in difficulty caused by mathematical forms that do not fall into the pattern of Eq. (7-1). This problem will appear in many of the performance equation derivations that occur later in this book.

CHAPTER 8

DIMENSIONAL ANALYSIS

DIMENSIONS IN PERFORMANCE EQUATIONS

8.1. Performance equation derivations follow more or less exactly the pattern described in Chapter 7, and lead to equations with the general form of Eq. (7-1). Equations of this kind represent physical situations in which the quantities involved may have any arbitrary dimensions. However, the mathematical form imposes certain restrictions on the performance equation coefficients, which are the a's and b's of Eq. (7-1). Certain of these restrictions are associated with the necessity for maintaining dimensional consistency among the terms of the equation. In addition to solving problems of this kind, a proper use of dimensional relationships may considerably decrease the work of obtaining results from performance equations and extending the application of these results to the widest possible range of engineering situations. The generalized study of applying the theory of dimensions to problems of this type is called Dimensional Analysis.

8.2. The raw material of dimensional analysis is supplied by performance equations with the general features of Eq. (7-1). In equations of this type, the dimensions of the input and output quantities are immaterial so long as the equation coefficients have the proper dimensions to make all individual terms of the equation dimensionally consistent with each other. For example, all of the terms of Eq. (7-1) may be divided by a_o to produce the form*

$$\frac{a_n}{a_o}\frac{d^n q_{out}}{dt^n} + \frac{a_{n-1}}{a_o}\frac{d^{n-1} q_{out}}{dt^{n-1}} + \dots + \frac{a_1}{a_o}\frac{dq_{out}}{dt} + q_{out} = \frac{b_o}{a_o}\left[\frac{b_{cnst}}{b_o} + q_{in} + \frac{b_1}{b_o}\frac{dq_{in}}{dt} + \dots + \frac{b_{m-1}}{b_o}\frac{d^{m-1}q_{in}}{dt^{m-1}} + \frac{b_m}{b_o}\frac{d^m q_{in}}{dt^m}\right] \qquad (8\text{-}1)$$

in which all terms must have the dimensions of the output quantity.

8.3. The symbols in any specific example of Eq. (8-1) all represent some physical quantity that is important in the action of the operating system with which the equation is associated. Actually, the situation is less complicated than appears from an inspection of the equation. The ratio of b_o to a_o is a dimensional similarity factor in the sense that it will always have the dimensions required to match the dimensions of the input quantity to the dimensions of the output quantity. With the exception of the ratio of the constant term to b_o, which has the dimensions of the input quantity, all the other coefficient ratios must have the dimensions of some power of time. This is true because the numerators of the derivative forms have only the dimensions of the input quantity or the output quantity, so that the coefficient ratio for any given term must have the dimensions of the proper power of time to balance out to zero the dimensions of time in the denominators of the derivatives.

---

* In general, the coefficients must be positive for instrument application.

WORKING FORM OF THE GENERALIZED PERFORMANCE EQUATION

8.4. Because of the part played by the dimensional similarity factor in matching dimensions on both sides of the performance equation form of Eq. (8-1), and the fact that all coefficient ratios have the dimensions of some power of time, the essential mathematical features of the generalized equation are completely independent of input and output dimensions. This means that all the analytical manipulation required to find the output response corresponding to given input variations may be carried out in terms of a working form of the performance equation in which the terms are made up of working variables that are completely divorced from any connection with specific physical situations. The working form of the generalized performance equation is:

$$a'_n \frac{d^n v}{dt^n} + a'_{n-1} \frac{d^{n-1} v}{dt^{n-1}} + \dots + v = \sigma[b'_{cnst} + u + b'_1 \frac{du}{dt} + \dots + b'_m \frac{d^m u}{dt^m}] \qquad (8\text{-}2)$$

where the working variable symbols corresponding to the physical quantity symbols of Eq. (8-1) are

$q_{out} \approx v$ = dependent variable

$q_{in} \approx u$ = independent variable

$a'_n \approx \frac{a_n}{a_o}$ = n^{th} dependent variable equation coefficient ratio

$b'_m \approx \frac{b_m}{b_o}$ = m^{th} independent variable equation coefficient ratio

$\sigma \approx \frac{b_o}{a_o}$ = dimensional similarity factor

8.5. All of the theoretical analysis of Volume II is carried out in terms of working variables and the working variable form of the generalized performance equation. The working variables are not uniquely associated with any particular equation, but merely hold places in the functional relationship represented by any given form. For this reason, any one of the working variables may have many different meanings when theoretical results expressed in terms of working variables are applied to practical situations.

PROBLEMS OF DIMENSIONAL ANALYSIS

8.6. Dimensional analysis deals with the problems of organizing relationships among the equation coefficients, working variables, input quantities, and output quantities to realize the greatest useful return from theoretical and experimental studies of operating system performance. The discussion of this chapter develops the theoretical background of dimensional analysis and ends with an illustrative example.

PHYSICAL QUANTITIES AND MEASUREMENT

8.7. Physical quantities are things or states of things that can be described by means of numbers. The number associated with a given example of a physical quantity is the measure of the physical quantity. Mass, length, time, pressure, temperature, etc., can all be described in numerical terms, and thus fall into the class of physical quantities. In general, the symbols that represent physical quantities in equations actually refer to the measures of these quantities, not to the quantities themselves.

8.8. Each measurement always involves two ideas: first, the concept of the physical quantity to be measured; and second, the rule by which numbers are associated with particular states of the physical quantity. Thus the measurement of distance between two points involves the concept of length and a rule of measurement. This rule is that the measure of the given distance is the number of times a unit length can be fitted into that distance.

8.9. The number which measures the separation of two points depends on the size of the length unit. On the other hand, if one distance is twice as great as another distance, the measure of the first distance must be twice as great as the measure of the second distance regardless of the size of the length unit used. The general rule is that

> *ratios of the measures associated with a given situation that can be described in terms of a single physical quantity must be independent of the size of the unit used for the measurements. This requirement is fulfilled if the number which is the measure of a given quantity is inversely proportional to the size of the unit in which the measure is expressed.*

PRIMARY AND SECONDARY QUANTITIES

8.10. Primary quantities are fundamental in the sense that they can be measured only in terms of their own units. The theory of dimensions shows that various sets of quantities can be used as primary quantities. In practice, primary quantities are chosen in such a way that the ideas of dimensional analysis can be applied in the simplest possible fashion. Mass, length, time, and temperature are usually chosen as the primary quantities for engineering work. It is a matter of common experience that distance cannot be measured in mass units or time units, but only in units of length. This same statement is also true of the other three quantities and their individual units.

8.11. Secondary quantities are derived from primary quantities according to some rule of combination. Thus the velocity of an object moving uniformly between two points is the distance between the points divided by the time required for the object to move from one point to the other.

RULES OF COMBINATION

8.12. Rules of combination available for use in forming secondary quantities from primary quantities are restricted by the requirement that ratios of measures associated with concrete examples of the same secondary quantity must be independent of the size used for the primary units. For example, the ratio of the average speeds of two automobiles moving over a given distance in ten seconds and twenty seconds, respectively, is always two no matter what primary units of length and time are used in finding the measures of the speeds. Bridgman(4) shows by mathematical reasoning that in any particular case the measure of a secondary quantity, S, must depend upon the measures $P_1, P_2 \ldots P_m$ of the m primary quantities which may be involved in a way that fulfills an equation of the form

$$S = CP_1^{p_1} P_2^{p_2} \ldots P_m^{p_m} \tag{8-3}$$

The symbol C of Eq. (8-3) in general is a pure number (i.e., it is dimensionless). The exponents $p_1, p_2 \ldots p_m$ are called the dimensions of the given secondary quantity in the primary quantities $P_1, P_2 \ldots P_m$. In words, Eq. (8-3) states that

> *every secondary quantity must be expressible as some number multiplied by powers of the measures of the primary quantities.*

The concept of dimensions is useful in the problems of engineering. Many examples demonstrating the truth of this statement are included in this book.

8.13. The manner in which Eq. (8-3) keeps the ratio of secondary quantities independent of the size of the primary units in any particular situation is easily demonstrated by actual substitutions. A change in the size of the primary units introduces the same numerical factors in each one of the measures associated with a series of concrete examples of the secondary quantity. These numerical factors will always cancel out when ratios of the measures of the secondary quantity are taken.

THE Π THEOREM

8.14. The fundamental ideas of dimensional analysis are used to some extent by all persons who must analyze physical situations. For example, checks of equations to make sure that all the terms involved have proper dimensions are almost instinctively carried out. In a similar fashion, experience soon teaches the advantages of making plots in terms of nondimensional parameters which show characteristic shapes without being affected by changes in units or in magnitudes of the variables involved.

8.15. Bridgman(4) gives a rigorous mathematical derivation of a theorem first explicitly stated and called the Π Theorem by Buckingham.(6) This theorem concisely summarizes the possibilities and limitations of dimensional analysis as applied to physical problems. A brief discussion of the Π Theorem and its application to operating systems is given below.

8.16. Any physical situation which may be analyzed by mathematical reasoning can be described by means of an equation in which the physical quantities involved are the variables. Rigorously it is the measures of the physical quantities which appear in the equation rather than the variables themselves. The essential physical variables will be either primary quantities or secondary quantities, and each must therefore have the dimensional form of a product of powers of the primary quantities.

8.17. In addition to the physical variables, the arguments of the equation which describes a given situation may include dimensional constants which do not vary with physical changes, but depend upon the nature and size of the primary units selected for use in the problem. These constants are necessary to make the equations dimensionally correct and are similar to secondary quantities in that they have the form of products of powers of the primary quantities. The constant of universal gravitation which appears in Newton's law for the attraction between bodies is an example of a dimensional constant.

8.18. The relationship between the n physical variables and dimensional constants required to describe a set of physical facts can be expressed mathematically by an equation of the form

$$\psi(S_1, S_2 \ldots S_n) = 0 \qquad (8\text{-}4)$$

To be strictly true, Eq. (8-4) must contain all the essential variables which affect the situation represented by the equation. To describe any physical situation, which will certainly not be influenced in any way by changes in the size of primary units, Eq. (8-4) must remain unchanged no matter how the units of the primary quantities are changed in size. If this second condition is fulfilled, the equation is said to be a complete equation. The discussion which follows will be limited to equations of this type.

8.19. The n arguments of Eq. (8-4) will change in magnitude if the primary units are changed in size. However, if the equation is complete, such changes must be without effect on the equation itself. This condition will be fulfilled if the n arguments $S_1, S_2 \dots S_n$ are combined into new arguments $\Pi_1, \Pi_2 \dots$, each of which is dimensionless in the primary quantities. These new arguments are all made up of quantities which have the dimensional nature of secondary quantities and so must themselves have the form of products of powers of the primary quantities combined in such a way that each of the Π's is dimensionless in every one of the primary quantities. Under these circumstances, changes in the size of any or all of the primary units will not change the Π's, and consequently cannot affect Eq. (8-4), which can then be transformed into terms of dimensionless arguments as

$$\phi(\Pi_1, \Pi_2 \dots) = 0 \tag{8-5}$$

With the existence of the "Π" equation demonstrated, it remains to find the number and general form of the Π's.

8.20. If m primary quantities $P_1, P_2 \dots P_m$ are involved, the n variables of the original equation must be expressible as

$$\begin{aligned}
S_1 &= P_1^{p_{11}} P_2^{p_{21}} \dots P_m^{p_{m1}} \\
S_2 &= P_1^{p_{12}} P_2^{p_{22}} \dots P_m^{p_{m2}} \\
&\vdots \\
S_n &= P_1^{p_{1n}} P_2^{p_{2n}} \dots P_m^{p_{mn}}
\end{aligned} \tag{8-6}$$

where $p_{11} \dots p_{mn}$ are the dimensions of the secondary quantities in the primary quantities. The definition of each secondary quantity is known and hence the exponents $p_{11} \dots p_{mn}$ are known. Each one of the Π's has the general nature of a secondary quantity and for this reason must have the form of a product of powers of the secondary quantities $S_1 \ldots S_n$. This means that

$$\begin{aligned}
\Pi_1 &= S_1^{s_{11}} S_2^{s_{21}} \ldots S_n^{s_{n1}} \\
\Pi_2 &= S_1^{s_{12}} S_2^{s_{22}} \ldots S_n^{s_{n2}} \\
&\cdot \\
&\cdot \\
&\cdot \\
\Pi_n &= S_1^{s_{1n}} S_2^{s_{2n}} \ldots S_n^{s_{nn}}
\end{aligned} \tag{8-7}$$

In order to find the form of the Π's, the exponents $s_{11}, s_{21} \ldots$ must be determined. The generalized dimensionless variable Π_i can be written in the form

$$\begin{aligned}
\Pi_i &= S_1^{s_{1i}} S_2^{s_{2i}} \ldots S_n^{s_{ni}} \\
&= (P_1^{p_{11}} P_2^{p_{21}} \ldots P_m^{p_{m1}})^{s_{1i}} (P_1^{p_{12}} P_2^{p_{22}} \ldots P_m^{p_{m2}})^{s_{2i}} \ldots (P_1^{p_{1n}} P_2^{p_{2n}} \ldots P_m^{p_{mn}})^{s_{ni}}
\end{aligned} \tag{8-8}$$

To make Eq. (8-8) dimensionless in the primary quantities, it is necessary and sufficient to determine values for the exponents $s_{1i} \ldots s_{ni}$ in such a way that the sum of the exponents of each one of the primary quantities is equal to zero. The required conditions are stated mathematically by the equations

$$\begin{aligned}
p_{11} s_{1i} + p_{12} s_{2i} + \cdots p_{1n} s_{ni} &= 0 \\
p_{21} s_{1i} + p_{22} s_{2i} + \cdots p_{2n} s_{ni} &= 0 \\
&\cdot \\
&\cdot \\
&\cdot \\
p_{m1} s_{1i} + p_{m2} s_{2i} + \cdots p_{mn} s_{ni} &= 0
\end{aligned} \tag{8-9}$$

Equation (8-9) constitutes a set of m algebraic equations in n unknowns which can be solved by methods discussed in any text on advanced algebra. The mathematical theory of such a system of equations shows that (n-m) of the unknowns must be arbitrarily chosen in order for the remaining m unknowns to be uniquely determined. In the present case, this means that (n-m) of the exponents $s_{1i} \ldots s_{ni}$ may be arbitrarily chosen. After such a set of (n-m) choices is made, Eq. (8-9) can be solved for the remaining m factors, and the dimensional form of one of the Π's is determined. The theory of linear equations shows that it is possible to arbitrarily choose (n-m) independent sets of the variables in making the choices discussed above. Each of these (n-m) independent sets corresponds to a different Π. It follows that there will be a total of (n-m) dimensionless arguments for Eq. (8-5) which are independent in the sense that no one of them can be derived from any other through direct multiplication by some constant.

8.21. The conclusions from the derivation outlined above may be summarized in the Π Theorem. This theorem states:

> *if a physical situation can be represented by means of a complete equation,*
>
> $$\psi(S_1, S_2 \ldots S_n) = 0$$
>
> *where the n arguments include all the primary quantities, secondary quantities, and dimensional constants which must be considered in the problem, the equation can be written in the form*
>
> $$\phi(\Pi_1, \Pi_2 \ldots \Pi_{n-m}) = 0$$
>
> *in which* $\Pi_1, \Pi_2 \ldots \Pi_{n-m}$ *are the* $(n - m)$ *independent products of the arguments* $S_1 \ldots S_n$, *which are dimensionless in the m primary quantities.*

USE OF THE Π THEOREM

8.22. The Π Theorem is chiefly valuable because it directs attention to the fact that any physical problem can be studied in terms of dimensionless ratios. It also specifies the number of these ratios which are independent in any given case. The theorem itself does not prescribe any method for finding the Π's, but a study of the derivation suggests a formal mathematical procedure that may be used. This procedure does not greatly simplify the general problem of finding the Π's because an infinite number of choices is possible for each of the (n-m) exponents which must be arbitrarily selected before any one of the Π's can be determined. The usefulness of any set of Π's depends upon the skill exercised in choosing the arbitrary exponents. As a general rule, the Π's should be of such a nature that the solutions which describe the physical situation are in the most useful mathematical forms. In the absence of direct experience, the best Π's to use must depend upon a process of trial and error.

8.23. It is reasonable in any case to select one of the Π's so that it includes the dependent variable, while another Π should include the independent variable. For example, if the response of an engine indicator to a rapid pressure change is under consideration, one Π should contain the scale reading in terms of pressure, while another should contain the applied pressure. Only one Π should be formed with the dependent variable in order that an explicit solution for this variable may be written. Time is generally important as the running variable in dynamic problems, so that one of the Π's should contain time in combination with some interval associated with either the manner in which the applied pressure changes or with some characteristic of the instrument itself which is describable by a term that contains time as a factor. Other Π's should contain only the variables which describe the essential properties of the instrument involved. The process of selecting Π's must continue until each one of the n arguments appears at least once in one of the Π's.

8.24. Ordinarily a number of combinations among the arguments suitable for Π's can be selected by inspection, especially in the cases of the dependent and independent variables. On the other hand, suitable combinations of performance equation coefficients are sometimes difficult to find. A good method is to select unity as the exponent of the quantity which is of most interest while $(n - m - 1)$ of the other exponents are arbitrarily made equal to zero. This procedure results in a combination of $(m + 1)$ exponents of which m are to be determined. These unknown exponents can now be determined by setting up m algebraic equations in a fashion similar to that used in the derivation of the Π Theorem and solving these equations for the m unknowns. An example in which a set of Π's is determined by this procedure is discussed in following paragraphs.

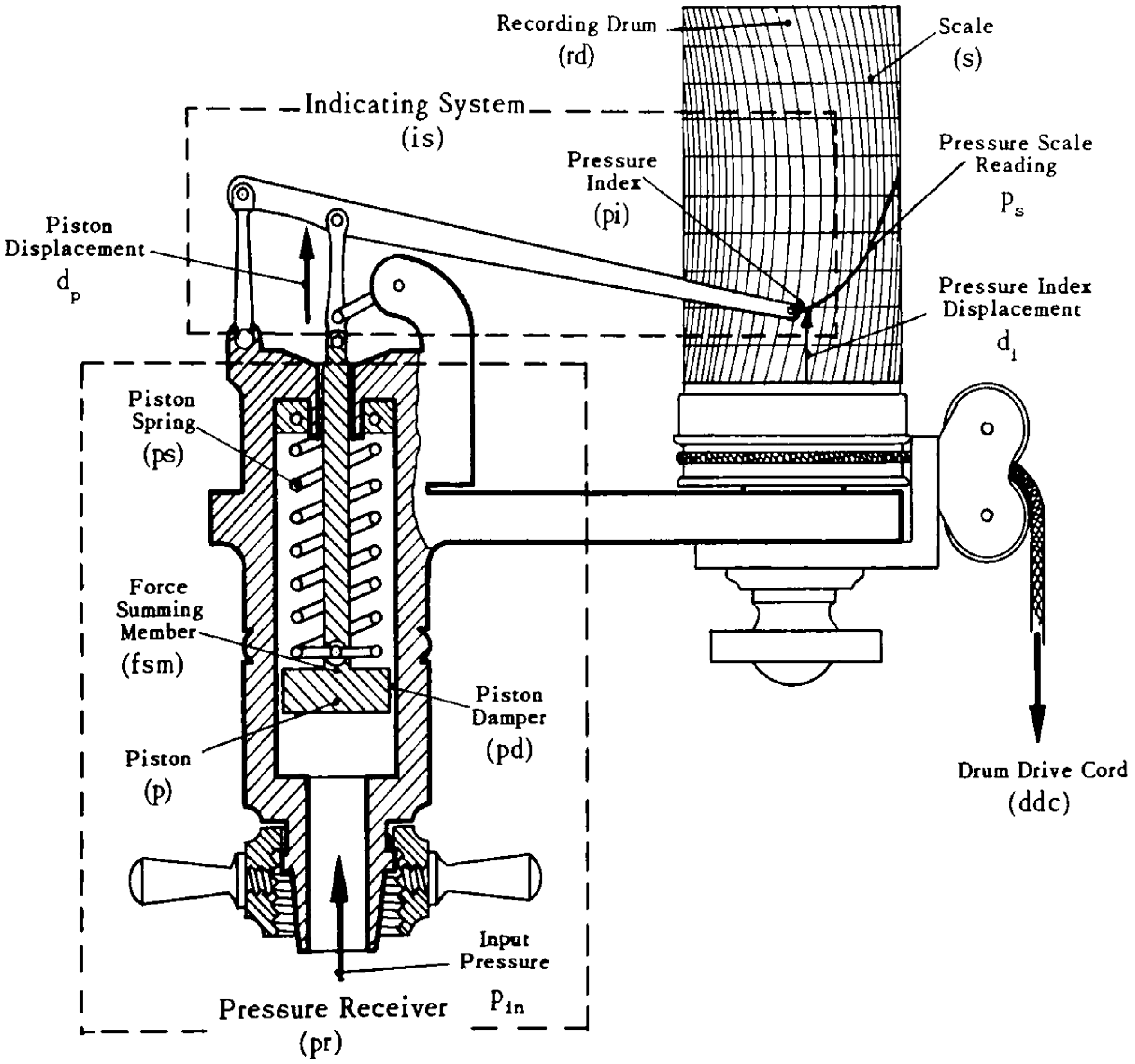


Fig. 8-1. Pictorial diagram for the spring-and-piston-type engine indicator.

SPRING-AND-PISTON ENGINE INDICATOR PERFORMANCE AS AN EXAMPLE

8.25. The spring-and-piston* engine indicator is taken as an example to illustrate the development of a typical performance equation and the corresponding working variable form. These equations are then used to demonstrate the use of dimensional analysis in finding combinations of equation coefficients and variables for effectively describing the behavior of an actual operating system.

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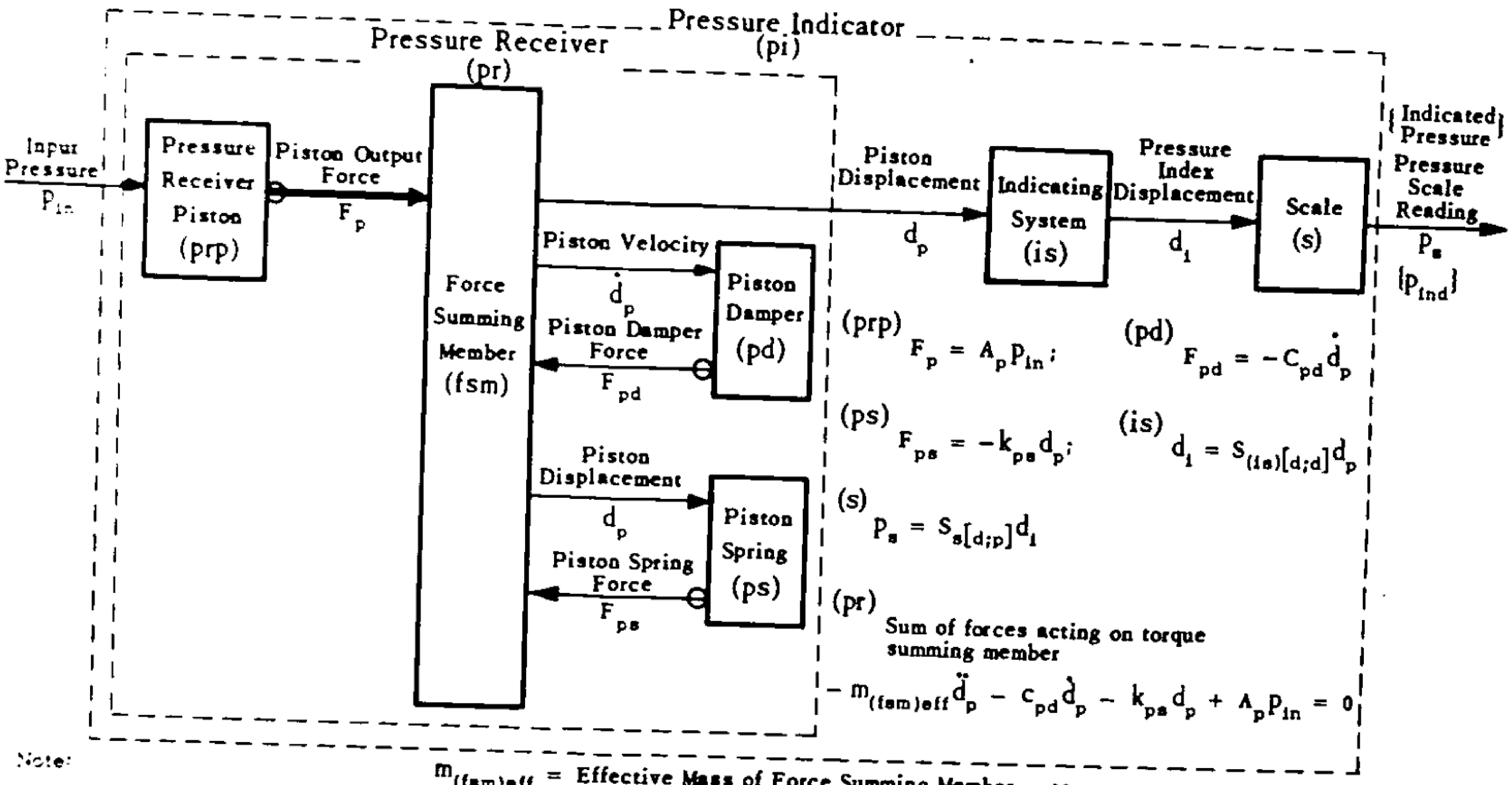
* Engine indicators of the spring-and-piston type were developed for the relatively slow pressure variations of the reciprocating piston steam engine. For this reason, instruments with the features of Fig. 8-1 are commonly called steam engine indicators.

8.26. Figure 8-1 is a pictorial diagram of an engine indicator of the spring and piston type. A closely fitting piston slides in a cylinder which is open at one end to the input pressure. A rod connected to the piston, and the piston itself, form a rigid body which acts as the force summing member. The piston spring acts in compression between the cylinder and the force summing member to resist the force acting on the piston due to the input pressure. The small clearance between the piston and cylinder is filled with a film of oil and acts as the piston damper. The piston displacement acts as the input to a four-bar linkage which serves as the indicating system. The output of the indicating system is the pressure index displacement. The scale is marked* on recording paper that is attached to the recording drum. This drum is positioned in rotation by the drum drive cord, which is made to move with the engine piston position by means of a driving linkage not shown in Fig. 8-1.



Note:

Mass effects due to indicating system parts that move with the force summing member are taken into account by the effective mass chosen for the force summing member.

$m_{(fsm)eff}$ = Effective Mass of Force Summing Member

c_{pd} = Piston Damper Damping Coefficient

k_{ps} = Piston Spring Elastic Coefficient

A_p = Pressure Receiver Piston Area

Note:

A dot over a symbol represents differentiation with respect to time, i.e., $(\dot{\ }) \approx \frac{d}{dt}$; $(\ddot{\ }) \approx \frac{d^2}{dt^2}$

Fig. 8-2. Functional diagram for the spring-and-piston engine indicator.

FUNCTIONAL DIAGRAM FOR THE SPRING-AND-PISTON ENGINE INDICATOR

8.27. Figure 8-2 is a functional diagram for the engine indicator of Fig. 8-1. This diagram has defining names and the corresponding letter symbols placed within the boxes that represent operating components. The operating components that receive input pressure and produce piston displacement are enclosed in a dashed-line box and designated as the pressure receiver, with the symbol (pr). The name pressure indicator, with the symbol (pi), is used to represent the complete system which receives input pressure and produces scale reading as its output. Figure 8-2 shows that the pressure indicator is functionally an open-chain arrangement with the direct quantity flow going from the pressure receiver to the indicating system to the scale. A summary of performance equations for these chain links and the pressure receiver operating components is given at the right-hand side of Fig. 8-2. Word definitions for symbols appearing in the performance equation summary and not included among the quantities shown on the diagram are listed in the figure.

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* In practice, the recording paper does not usually carry graduations. Indicated pressure readings are associated with index displacements by means of rulers marked off with proper units.

DERIVATION OF ENGINE INDICATOR PERFORMANCE EQUATION

8.28. Derivation Summary 8-1 shows the development of the engine indicator performance equation from the performance equations for the three operating chain links of Fig. 8-2. This derivation summary also illustrates the relationships between the engine indicator performance equation and the corresponding generalized performance form of Eq. (7-1). The transformations required to set up the working variables and the working variable equation from the performance equation are also outlined in Derivation Summary 8-1. In practice, all theoretical operations are carried out with the working variable equation of Derivation Summary 8-1 in generalized terms. By using properly chosen nondimensional variables for these operations, the results appear in forms that may be immediately applied to a range of problems by simple use of transformations like those of Eq. (7) of Derivation Summary 8-1. In addition to extending the usefulness of results, suitable nondimensional variables facilitate analysis by suggesting effective behavior-describing concepts and allowing computations to be carried out in terms of the simplest possible numbers. For example, it is good practice to choose working variables that vary in the region of unity for the operating range of primary interest. It is the function of dimensional analysis to assist in the choice of variables for use in finding solutions to the problems of instrument engineering.

PERFORMANCE DESCRIPTION

8.29. Performance description for any given operating system must be based to at least some extent on the performance equation of the system. It is true that the performance description for an operating component might be based entirely on experimental data; however, it is also true that the effectiveness of any given amount of information on a situation of even moderate complexity will usually be greatly increased if the performance equation is considered when the pattern for (pr) + (is) Pressure Receiver and Indicating System

$$m_{(fsm)(eff)}\,\ddot{d}_i + c_{pd}\,\dot{d}_i + k_{ps}\,d_i = A_p\,S_{(is)[d;d]}\,p_{(in)} \tag{1}$$

(ei) Engine Indicator, equivalent to
(pr) + (is) + (s) Pressure Receiver and Indicating System and Scale

$$m_{(fsm)(eff)}\,\ddot{p}_s + c_{pd}\,\dot{p}_s + k_{ps}\,p_s = A_p\,S_{(is)[d;d]}\,S_{s[d;p]}\,p_{in} \tag{2}$$

Equation (2) is the performance equation for the engine indicator. This equation corresponds to the generalized performance equation form:

$$a_2\,\ddot{q}_{out} + a_1\,\dot{q}_{out} + a_0\,q_{out} = b_0\,q_{in} \tag{3}$$

The symbol equivalence between equations (3) and (2) is

$$a_2 \approx m_{(fsm)(eff)} \qquad a_1 \approx c_{pd} \qquad a_0 \approx k_{ps} \qquad b_0 \approx A_p\,S_{(is)[d;d]}\,S_{s[d;p]} \tag{4}$$

When both sides of equation (2) are divided by k_{ps}, the result is

$$\frac{m_{(fsm)(eff)}}{k_{ps}}\,\ddot{p}_s + \frac{c_{pd}}{k_{ps}}\,\dot{p}_s + p_s = S_{(ei)[p;p]}\,p_{in} \tag{5}$$

where

$$S_{(ei)[p;p]} = \frac{A_p\,S_{(is)[d;d]}\,S_{s[d;p]}}{k_{ps}}$$ ≡ Pressure Input – Pressure Scale Reading Output Sensitivity of the Engine Indicator

Note: Under static and quasi-static conditions an ideal engine indicator will operate so that $p_s = p_{in}$
This means that for an ideal pressure indicator $S_{(ei)[p;p]} = 1$

Equation (5) corresponds to the generalized form of equation (3), which is

$$\frac{a_2}{a_0}\,\ddot{q}_{out} + \frac{a_1}{a_0}\,\dot{q}_{out} + q_{out} = \frac{b_0}{a_0}\,q_{in} \tag{6}$$

Equation (6) becomes the working variable form of equation (5) if the generalized coefficients are given the following meanings in terms of the particular coefficients of the engine indicator performance equation:

$$\frac{a_2}{a_0} \approx \frac{m_{(fsm)(eff)}}{k_{ps}} \approx a_2' \qquad \frac{a_1}{a_0} \approx \frac{c_{pd}}{k_{ps}} \approx a_1' \qquad a_0' = 1 \qquad \frac{b_0}{a_0} \approx S_{(ei)[p;p]} \approx \sigma$$

and

$$q_{out} \approx p_s \approx v \qquad q_{in} \approx p_{in} \approx u \tag{7}$$

The working variable form corresponding to the engine indicator performance equation is a special case of equation (3), i.e.,

$$a_2'\,\ddot{v} + a_1'\,\dot{v} + v = \sigma u \tag{8}$$

Note that all manipulations of analysis may be carried out with the working variable form. The results in terms of the working variable may be applied immediately to the operating system under consideration by the use of equations (7).

Derivation Summary 8-1. Illustrative example of the performance equation – working equation relationship for the engine indicator.

describing performance is established. The performance characteristics for an operating system specify functional properties of the system, but do not of themselves describe the output from the system in any particular situation. A specific description of this type becomes possible only if the input variation is given in quantitative terms and the conditions of the system output are known at the initial instant, which is the instant at which a discontinuity occurs in the input variation, or an instant when the operating system is allowed to start working after being inoperative for a finite interval of time. The curves of Fig. 8-3 illustrate the responses of an operating system to a step function input change in terms of engine indicator behavior as a typical example. Initial conditions are shown as the indicated pressure and the rate of change of indicated pressure when the input discontinuity occurs. The instant of the discontinuity is the initial instant, shown as $t = 0$ in Fig. 8-3, to be used in applying mathematical methods for finding response curves.

RESPONSE CURVE FAMILIES

8.30. Problems that involve finding the output response from an operating system of known characteristics to a given input variation when initial conditions are specified can be resolved by straightforward analysis. On the other hand, the design problem of determining the characteristics of individual operating components required to produce a specified performance from an operating system and the companion problem of finding the best performance from given operating components are most conveniently solved if families of response curves are available for engineering studies. Each typical input forcing function requires one or more families of these response curves. The curves of a single family differ from each other in the performance equation coefficients assumed for the computations. This means that for even a relatively simple operating system, a number of individual curves is required to give a reasonable survey of performance possibilities. In practice, this number may be considerably reduced by using properly selected combinations of the performance equation coefficients, instead of the coefficients themselves, as performance-describing parameters that are held constant as the running variables vary over the ranges required to describe given physical situations. The reasoning patterns of dimensional analysis are valuable in that they suggest useful forms for dimensionless parameters and variables to be applied in performance studies. Except for rare instances, dimensional analysis deals only with forms for parameters and variables. In general, it offers no information on either the performance equations or the solutions for these equations.

8.31. Figure 8-3 illustrates the general nature of a response curve family. The engine indicator of Fig. 8-1, with the second-order differential equation form of Eq. (2) of Derivation Summary 8-1 as its performance equation, is taken as a specific example. The input forcing function is taken as a step function for which the change from the input pressure initial level to the input pressure forcing level occurs as a discontinuity at the instant chosen as the initial instant t = 0. The engine indicator response is assumed to start at an initial indicated pressure not identical with the input pressure initial level. The initial rate of change of indicated pressure is assumed to have an arbitrary value. The plots of Fig. 8-3 are made with input pressure and indicated pressure as ordinates, and with time as abscissae. All curves on Fig. 8-3 are plotted for engine indicators with a constant effective force summing member mass and a constant piston damper coefficient. Each of the three illustrative curves represents the response for a single value of the elastic restraint coefficient of the piston spring. A complete survey of performance characteristic possibilities would require a separate family of curves

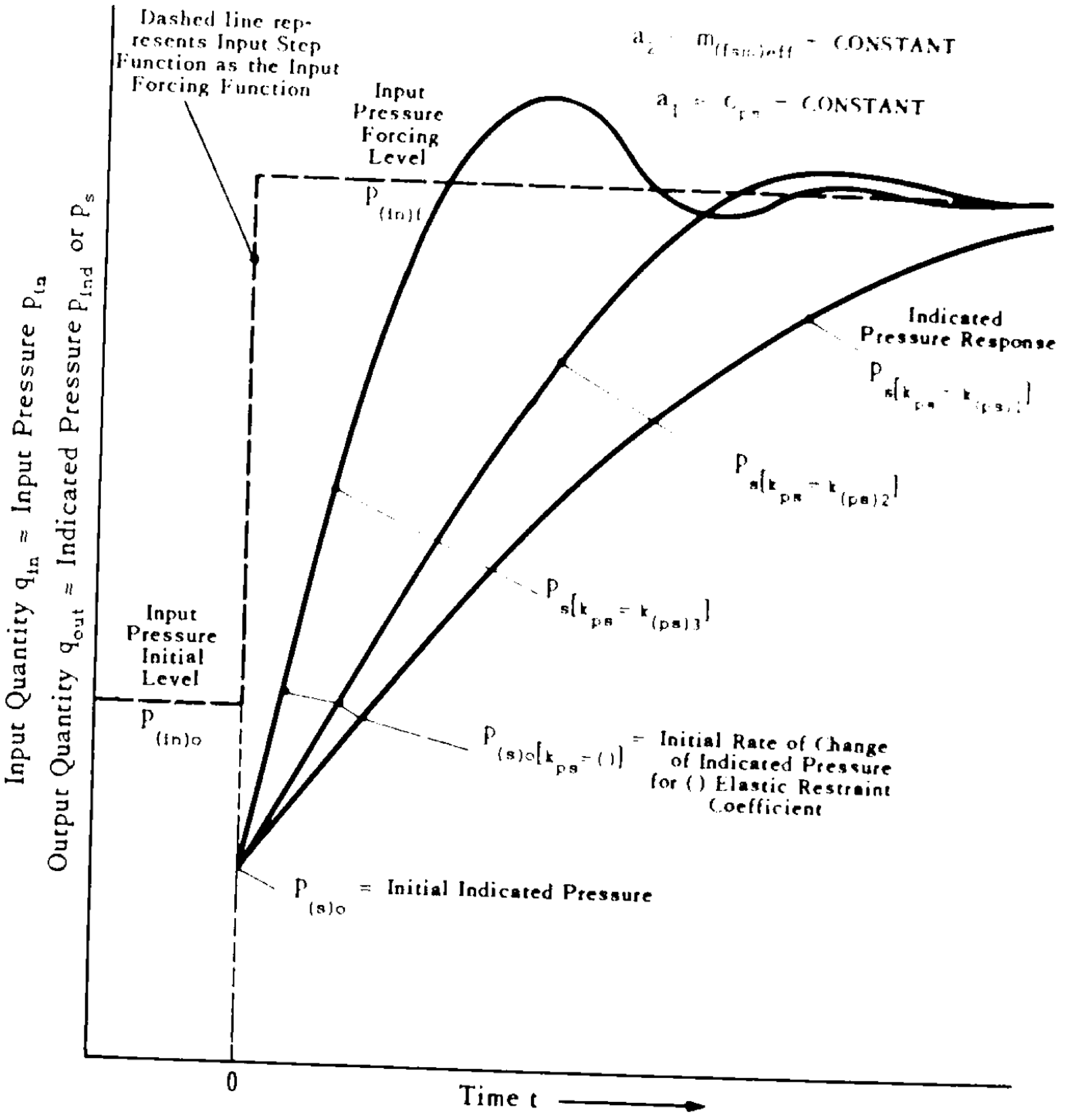
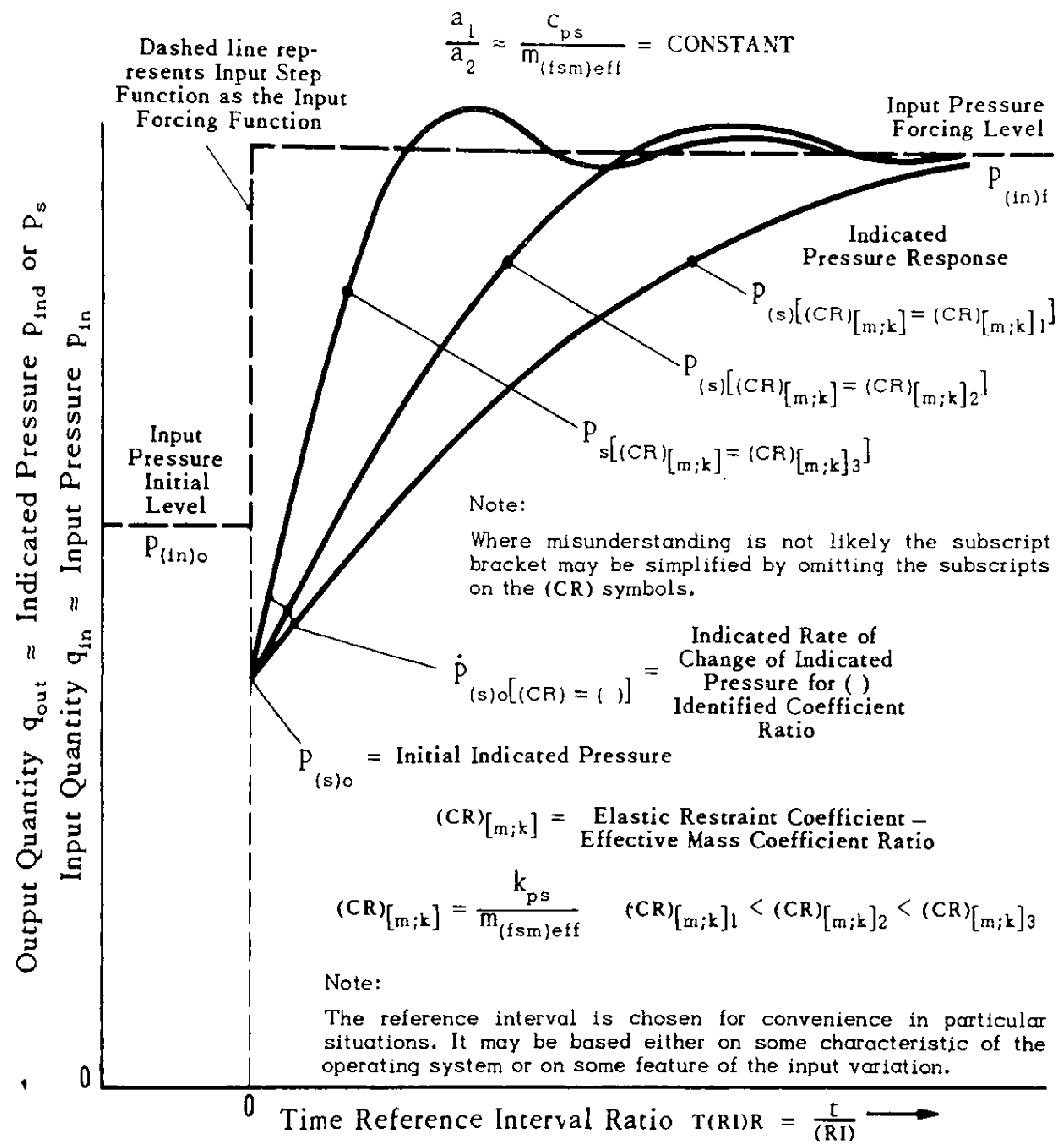


Fig. 8-3. Illustrative step function response of engine indicators with constant effective mass and damping coefficient for different values of the piston spring elastic restraint coefficient.

for each combination of values of the effective force summing member mass and a separate curve for each value of the elastic restraint coefficient.

8.32. The total number of separate curves required to survey completely the performance possibilities of the engine indicator may be reduced by using the ratio of the piston damper coefficient to the effective mass as the parameter for identifying each curve family, and the ratio of the elastic restraint coefficient to the effective mass as the parameter held constant for individual curves of a family. Figure 8-4 illustrates a curve family plotted with these ratios used as the parameters. Also, in this figure, the time scale has been made nondimensional by dividing time as a running variable by a reference interval. In Fig. 8-4 this reference interval is not specified. It is sufficient for present purposes to note that in practice a number of useful reference intervals often suggest themselves in the study of a problem. The selection of the best reference interval to employ in a



Note:
Where misunderstanding is not likely the subscript bracket may be simplified by omitting the subscripts on the (CR) symbols.

Note:
The reference interval is chosen for convenience in particular situations. It may be based either on some characteristic of the operating system or on some feature of the input variation.

Fig. 8-4. Illustrative step function responses of engine indicators with constant damping coefficient-effective mass ratio and different values of the elastic restraint coefficient-mass ratio.

given situation must depend upon judgment developed by experience in the field of operating system performance. The use of coefficient ratios to determine curve families like that of Fig. 8-4 reduces the number of curves required for a performance survey as compared with the number of curves required for the same survey if families similar to that of Fig. 8-3 are plotted. The truth of this statement depends upon the fact that when the coefficient ratios are used, they reduce by one the number of parameters to be assigned values in making up curve families. The coefficient ratios of Fig. 8-4 are not the most effective parameters for mapping the field of engine indicator performance, but they help to illustrate a general principle that, with the refinements made possible by use of the Π Theorem, greatly increases the effectiveness of performance-describing procedures.

ILLUSTRATIVE APPLICATION OF THE Π THEOREM

8.33. The engine indicator of Figs. 8-1 and 8-2 may be used as a typical example to illustrate the application of the Π Theorem to an actual case. Table 8-1 lists the names, symbols, and dimensional formulae for the physical quantities required to describe the response of the engine indicator to an input step function.

Table 8-1

Names, symbols and dimensional formulae for the variables required to describe the step function response of the engine indicator.

NAME	SYMBOL	DIMENSIONAL FORMULA
Input Pressure Forcing Level	$p_{(in)f}$	$ML^{-1}T^{-2}$
Input Pressure Initial Level	$p_{(in)o}$	$ML^{-1}T^{-2}$
Indicated Pressure	p_s	$ML^{-1}T^{-2}$
Initial Indicated Pressure	$p_{(s)o}$	$ML^{-1}T^{-2}$
Initial Rate of Change of Indicated Pressure	$p_{(s)o}$	$ML^{-1}T^{-3}$
Time	t	T
Effective Mass of Force Summing Member	$m_{(fsm)eff}$	M
Elastic Coefficient of Piston Spring	k_{ps}	MT^{-2}
Damping Coefficient of Piston Damper	c_{pd}	MT^{-1}

Note: The symbols **M**, **L** and **T** represent mass, length and time units, respectively, for dimensional formula purposes.

Note: The sensitivity $S_{(ei)[p;p]}$, which appears in Performance Equation (5) of Derivation Summary 8-1, is omitted from this table because of its dimensionless character.

Nine quantities appear in this table, that is, the n of the Π Theorem is nine. All of these quantities are dimensionally expressible in terms of mass (dimensional symbol M), length (dimensional symbol L), and time (dimensional symbol T). This means that the m of the Π Theorem is 3. For the Π Theorem, it follows that there are (9 – 3) = 6 independent dimensionless variables (Π's) for the step function response of the engine indicator. It is possible to reduce this number to 5 immediately if the height of the step function $p_{(in)f} - p_{(in)o}$ is taken as a convenient unit of pressure. Of the five Π's, it is reasonable to have one contain the indicated pressure as the dependent variable, while another includes the initial indicated pressure. These two Π's may be formed at once as

$$\Pi_1 = \frac{p_s}{p_{(in)f} - p_{(in)o}} \tag{8-10}$$

$$\Pi_2 = \frac{p_{(s)o}}{p_{(in)f} - p_{(in)o}} \tag{8-11}$$

8.34. It is necessary to provide a third Π with time as one of its factors. Indicated pressure and input pressure have both already been used in Π's and need not be considered in forming additional ratios. The initial rate of change of indicated pressure remains to be used, but it is logical to defer formation of a Π for this quantity until factors for nondimensionalizing time have been found.

8.35. The remaining variables that may be included in the Π containing t are m, c, and k. If the exponent of t is assumed to be unity, the Π can be written as

$$\Pi_3 = t m^{s_m} c^{s_c} k^{s_k} \tag{8-12}$$

8.36. The corresponding dimensional formula is

$$\Pi_3 = [T][M]^{s_m}[MT^{-1}]^{s_c}[MT^{-2}]^{s_k} \tag{8-13}$$

For Eq. (8-12) to be dimensionless, the sum of the exponents of each primary quantity must be equal to zero. With this combination of variables, there are three equations in two primary quantities. This makes it necessary to assume a value for one of these exponents in order to get a unique solution. Depending upon which exponent is arbitrarily assigned, there will be three different sets of values.

8.37. The procedure for finding a suitable set of exponents is to write the equations for the exponents in full and then assume each exponent to be zero in turn, thus

$$\begin{gathered} 1 - s_c - 2s_k = 0 \\ s_m + s_c + s_k = 0 \end{gathered} \tag{8-14}$$

First, if s_m is assumed to be zero, the solution of Eq. (8-14) is $s_c = -1$ and $s_k = 1$. This gives $t(k/c)$ for one Π. If s_c is assumed to be zero, the solution is $s_m = -1/2$, $s_k = 1/2$. This gives $t\sqrt{k/m}$ for the corresponding Π. Finally, if s_k is assumed to be zero, the solution of Eq. (8-14) is $s_m = -1$, $s_c = 1$, so the Π becomes $t(c/m)$. It follows that there is a choice of three different ratios which can be used as the Π containing t. It will be seen in succeeding chapters that each one of these three possibilities will have its particular usefulness, depending on the nature of the problem in which it appears. It is sufficient to note for the present that k/c, $\sqrt{k/m}$, and c m all have the dimension of T^{-1} .

8.38. A Π containing the initial rate of change of the scale reading $\dot{p}_{(s)o}$ can now be formed by inspection by noting that $\dot{p}_{(s)o}$ has the dimensions of a pressure divided by a time. Any one of the pressures and any one of the three combinations having the dimension of T^{-1} could be used. In the case of the pressure, however, it is reasonable to associate the Π with the initial scale reading $p_{(s)o}$. Using $p_{(s)o}$ in conjunction with each of the three combinations having the dimension of T^{-1} gives, as three choices for the Π,

$$\frac{\dot{p}_{(s)o}}{p_{(s)o}\frac{k}{c}}, \quad \frac{\dot{p}_{(s)o}}{p_{(s)o}\sqrt{\frac{k}{m}}}, \text{ and } \frac{\dot{p}_{(s)o}}{p_{(s)o}\frac{c}{m}}$$

Forms for four of the five possible Π's have now been found. It remains to find the fifth and last Π.

8.39. All the variables of Table 8-1 have been included in at least one Π. There are no new variables which must be included in the remaining Π. Dimensionless ratio Π's containing the independent variable, the dependent variable, and initial values of both displacement and velocity have been determined. However, a Π has not yet been found which contains only the design variables of the engine indicator. If the variables describing the indicator itself can be combined to form a Π, this Π will be uniquely associated with the indicator as an operating system. A Π of this kind can be found by writing

$$\Pi_5 = m^{s_m} c^{s_c} k^{s_k} \tag{8-15}$$

The corresponding dimensional formula is

$$\Pi_5 = [M]^{s_m}[MT^{-1}]^{s_c}[MT^{-2}]^{s_k} \tag{8-16}$$

The equations for the exponents are

$$\begin{aligned} s_m + s_c + s_k &= 0 \\ -s_c - 2s_k &= 0 \end{aligned} \tag{8-17}$$

If any one of these s's is placed equal to zero, no solution is possible. However, if any of the s's is placed equal to unity, solution gives a Π of the form $c/\sqrt{mk}$. This Π will appear later as the essential factor of a basic system parameter called the damping ratio.

8.40. Table 8-2 summarizes the ratios derived above to fulfill the Π Theorem requirements of five Π's as dimensionless variables suitable for use in describing engine indicator response to an input step function. The generalized reasoning of the Π Theorem shows that the engine indicator response fulfills a function of the type

$$\phi(\Pi_1, \Pi_2, \Pi_3, \Pi_4, \Pi_5) = 0 \qquad (8\text{-}18)$$

8.41. It is important to note that Eq. (8-18) does not give any information as to the form of the relationship involved. This information is provided by the performance equation derivation outlined in Derivation Summary 8-1.

8.42. The application of dimensional analysis discussed above is a simple, special case in comparison with situations that are common in instrument engineering. However, the problems that appeared in finding proper dimensionless ratios to describe engine indicator performance are similar to the problems to be solved in applications of dimensional analysis of situations of any complexity. The methods used in the illustrative example may be applied in any case. These methods will appear many times in the developments that occur in later chapters. The system of Π's generated in any particular case by the procedure illustrated above will always furnish a basic pattern for forming nondimensional quantities. Changes in this pattern will often be made to improve effectiveness in dealing with particular performance equation forms.

Table 8-2

Summary of dimensionless ratio variables for the description of engine indicator step function response.

$$\Pi_1 = \frac{P_s}{P_{(in)f} - P_{(in)o}}$$

$$\Pi_2 = \frac{P_{(s)o}}{P_{(in)f} - P_{(in)o}}$$

$$\Pi_3 = t\left(\frac{k}{c}\right), \text{ or } t\sqrt{\frac{k}{m}}, \text{ or } t\left(\frac{c}{m}\right)$$

$$\Pi_4 = \frac{\dot{P}_{(s)o}}{P_{(s)o}\left(\frac{k}{c}\right)}, \text{ or } \frac{\dot{P}_{(s)o}}{P_{(s)o}\sqrt{\frac{k}{m}}}, \text{ or } \frac{\dot{P}_{(s)o}}{P_{(s)o}\left(\frac{c}{m}\right)}$$

$$\Pi_5 = \frac{c}{\sqrt{mk}}$$

Note: To simplify the symbols, identifying subscripts have been omitted from the performance equation coefficients.

CHAPTER 9

CONCEPTS AND NOTATION FOR THE COMPARISON OF PHYSICAL QUANTITIES

LEVEL COMPARISONS

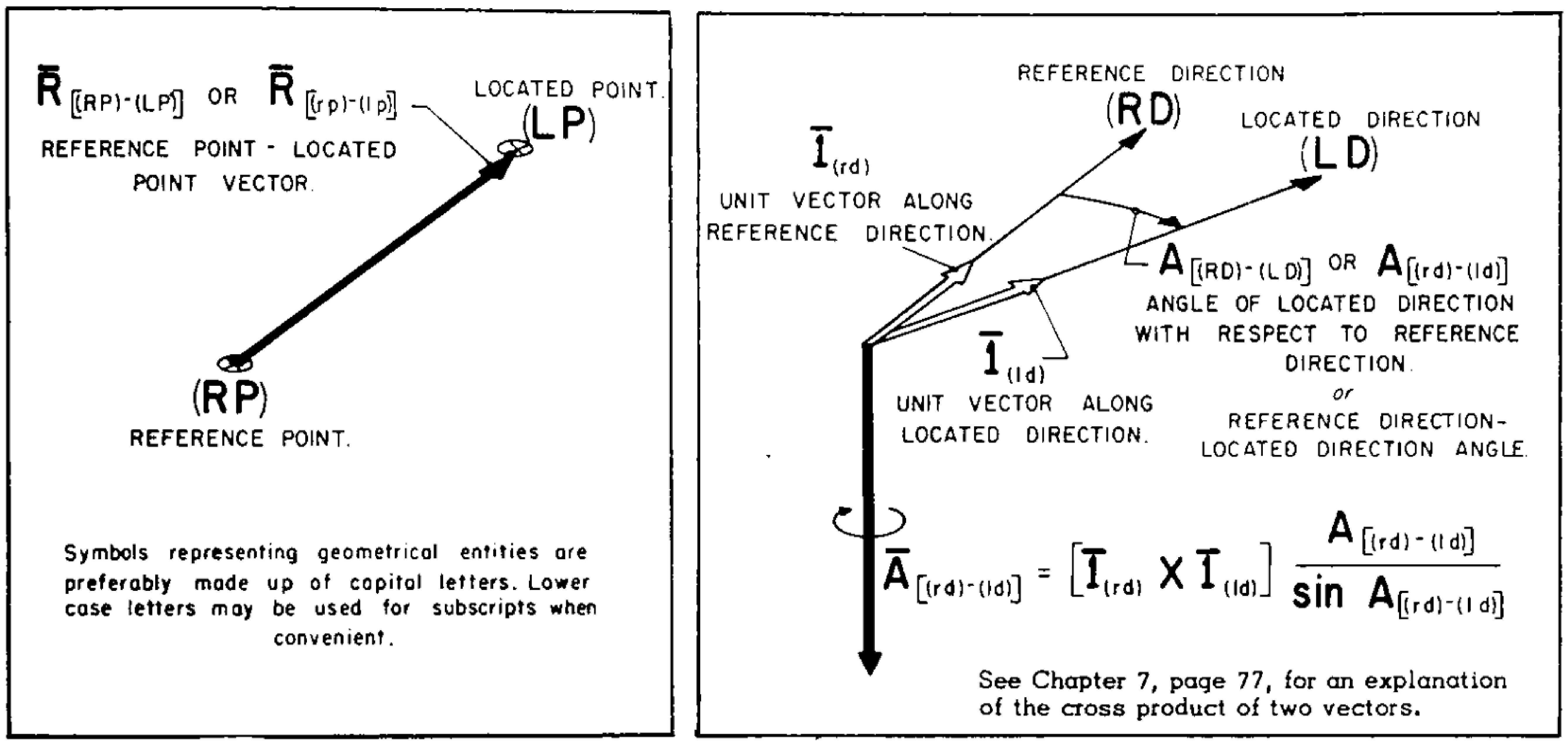
9.1. Instrument engineering ordinarily deals with operating systems of high performance in the sense that the outputs for given inputs bear a close relationship to some standard pattern. When the output alone is considered, good performance is often distinguished from poor performance by small percentage changes in large numbers that may be required to represent the output. For this reason it is generally more satisfactory in practice to establish some comparison level and deal with differences between levels of the quantity under study and this level than it is to deal directly with the output itself. These differences between the output as a compared level and the comparison level, which may be chosen either as constant or varying, may be given various names as suggested by the situations under consideration. In addition to the description of scalar differences between quantity levels, vector differences are important in geometrical applications where it is necessary to describe the difference in the position of a located point with respect to a reference point. This chapter includes definitions and symbols for location vectors of this type, and also for angular rotations, velocities and accelerations. Common patterns for these definitions and symbols to describe geometrical concepts and scalar difference concepts are developed and illustrated by a number of examples.

THE LOCATION VECTOR

9.2. Figure 9-1a illustrates the concept of the location vector that describes the position of a located point (symbol (LP)) with respect to a reference point (symbol (RP)) in terms of a vector for which the main symbol is $\bar{R}$ (the superscript bar shows that the "radius," R, is a vector quantity). The location vector symbol is completed with a subscript bracket containing two symbols. The subscript position nearest the main symbol is the comparison point symbol, or reference point symbol position, which is (RP) in Fig. 9-1a. The subscript position to the right of the reference point position is the compared point position and contains the symbol (LP), for "located point," in Fig. 9-1a. When it is convenient and no misunderstanding is likely to occur, lower-case letters may be substituted for the capital letters in the subscript of the location vector symbol.

9.3. Figure 9-1b illustrates the application of the concept of the reference quantity — compared quantity difference applied to the angle between two lines. In this figure the reference direction has the symbol (RD), and the located direction has the symbol (LD). The main symbol for the angle is A. This main symbol carries a reference direction — compared direction bracket which has (RD) in the reference quantity position and (LD) in the located quantity position.

9.4. In Fig. 9-1b, the angle is shown in terms of a vector representation in which it is expressed as the cross product of the unit vector along the reference direction and the unit vector along the located direction, multiplied by the ratio of the magnitude of the angle to the sine of the angle. It is to be noted that the order of subscripts in the reference quantity — compared quantity bracket is the same as the order of subscripts on the unit vectors of the cross product.


$\bar{R}_{[(RP)-(LP)]}$ OR $\bar{R}_{[(rp)-(lp)]}$
REFERENCE POINT - LOCATED POINT VECTOR
LOCATED POINT
(LP)
(RP)
REFERENCE POINT
REFERENCE DIRECTION
(RD)
LOCATED DIRECTION
(LD)
$\bar{1}_{(rd)}$
UNIT VECTOR ALONG REFERENCE DIRECTION
$A_{[(RD)-(LD)]}$ OR $A_{[(rd)-(ld)]}$
ANGLE OF LOCATED DIRECTION WITH RESPECT TO REFERENCE DIRECTION
or
REFERENCE DIRECTION-LOCATED DIRECTION ANGLE
$\bar{1}_{(ld)}$
UNIT VECTOR ALONG LOCATED DIRECTION
$\bar{A}_{[(rd)-(ld)]} = \left[\bar{1}_{(rd)} \times \bar{1}_{(ld)}\right] \frac{A_{[(rd)-(ld)]}}{\sin A_{[(rd)-(ld)]}}$

Symbols representing geometrical entities are preferably made up of capital letters. Lower case letters may be used for subscripts when convenient.

See Chapter 7, page 77, for an explanation of the cross product of two vectors.

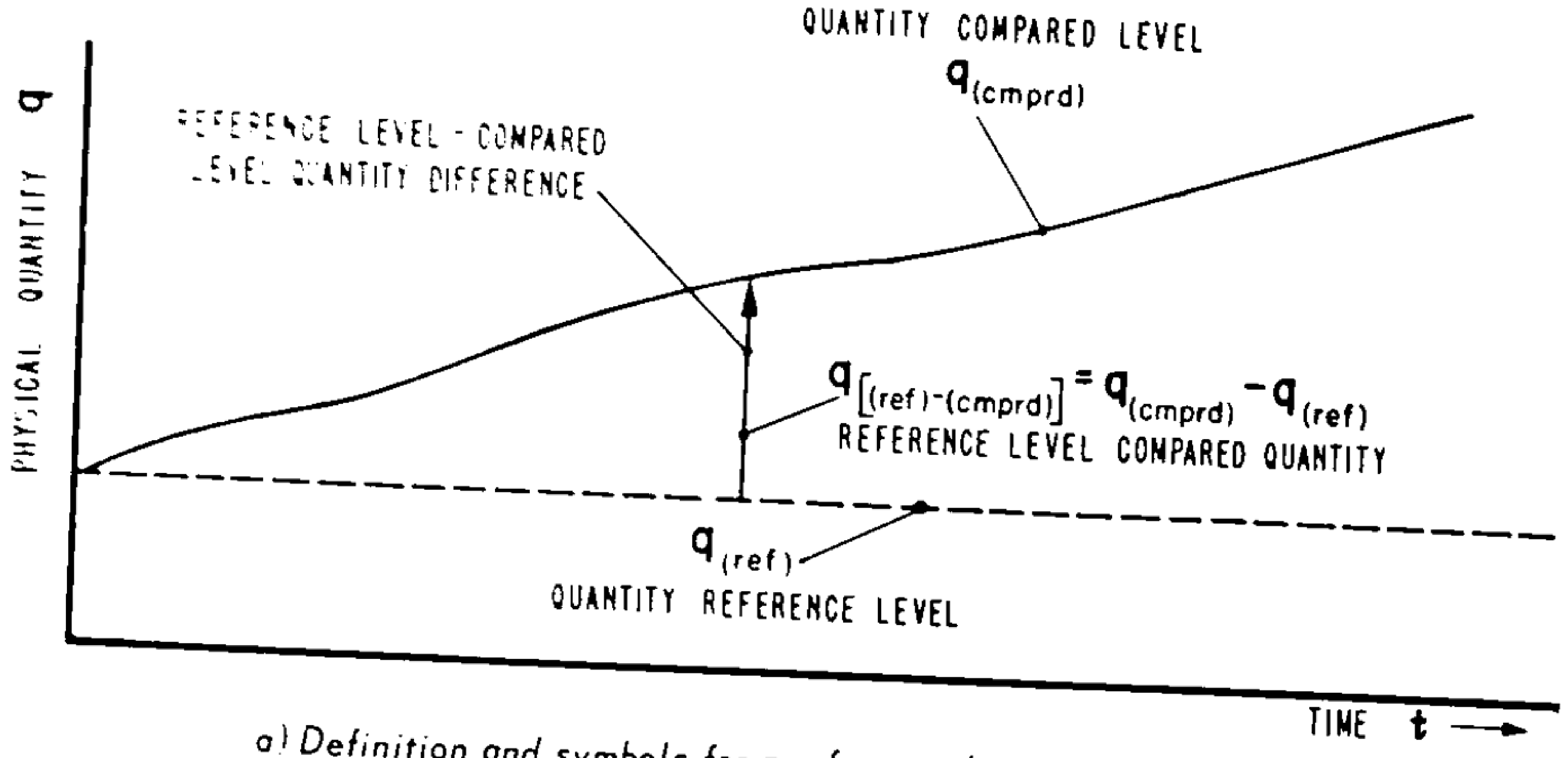
a) Vector representation for location vector from a located point with respect to a reference point

b) Planar and vector representation of angle between a reference direction and a located direction

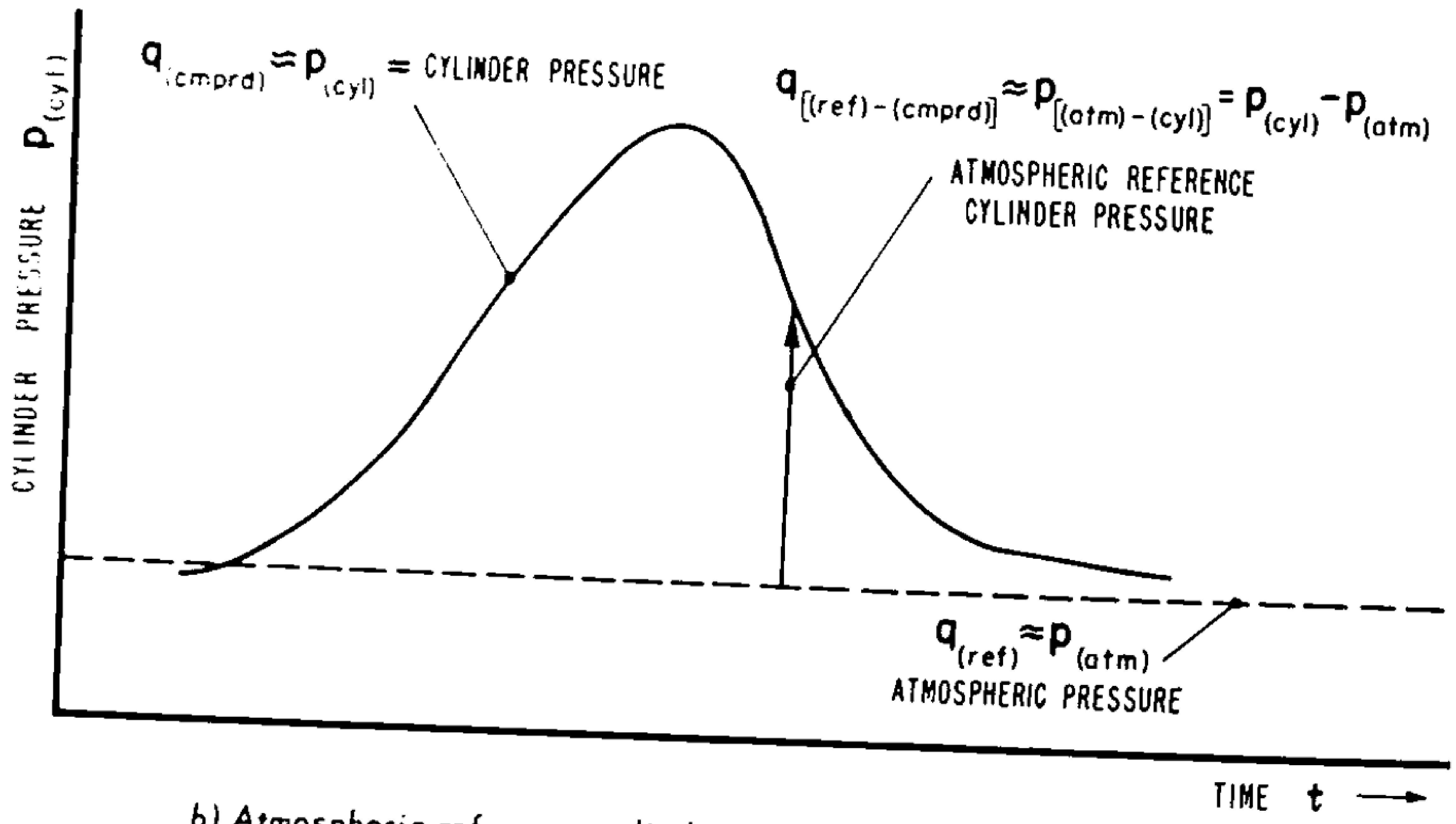
Fig. 9-1. Diagrams illustrating formation of symbols to represent the vector between two points and the angle between two lines.

SYMBOLS FOR REFERENCE LEVEL BASED QUANTITIES

9.5. Figure 9-2 illustrates the general concept of a reference level — compared level quantity difference when the quantity involved is a scalar (i.e., has magnitude and algebraic sign but no direction). In this figure the quantity reference level is shown as a constant, with the quantity compared level varying as a function of time. The reference level — compared level quantity difference has the main symbol q (for quantity). The symbol is completed by the reference level — compared level bracket, which contains (ref) (for reference level) in the comparison quantity position and (cmprd) (for compared level) in the compared quantity position. For convenience in use, the complete name of the quantity difference is shortened to reference level compared quantity. This name conveys all the necessary information and does not seriously encourage misunderstandings through omission of the word "difference" from its name.


QUANTITY COMPARED LEVEL
$q_{(cmprd)}$
REFERENCE LEVEL - COMPARED LEVEL QUANTITY DIFFERENCE
$q_{[(ref)-(cmprd)]} = q_{(cmprd)} - q_{(ref)}$
REFERENCE LEVEL COMPARED QUANTITY
$q_{(ref)}$
QUANTITY REFERENCE LEVEL
PHYSICAL QUANTITY q
TIME t

a) Definition and symbols for a reference level compared quantity



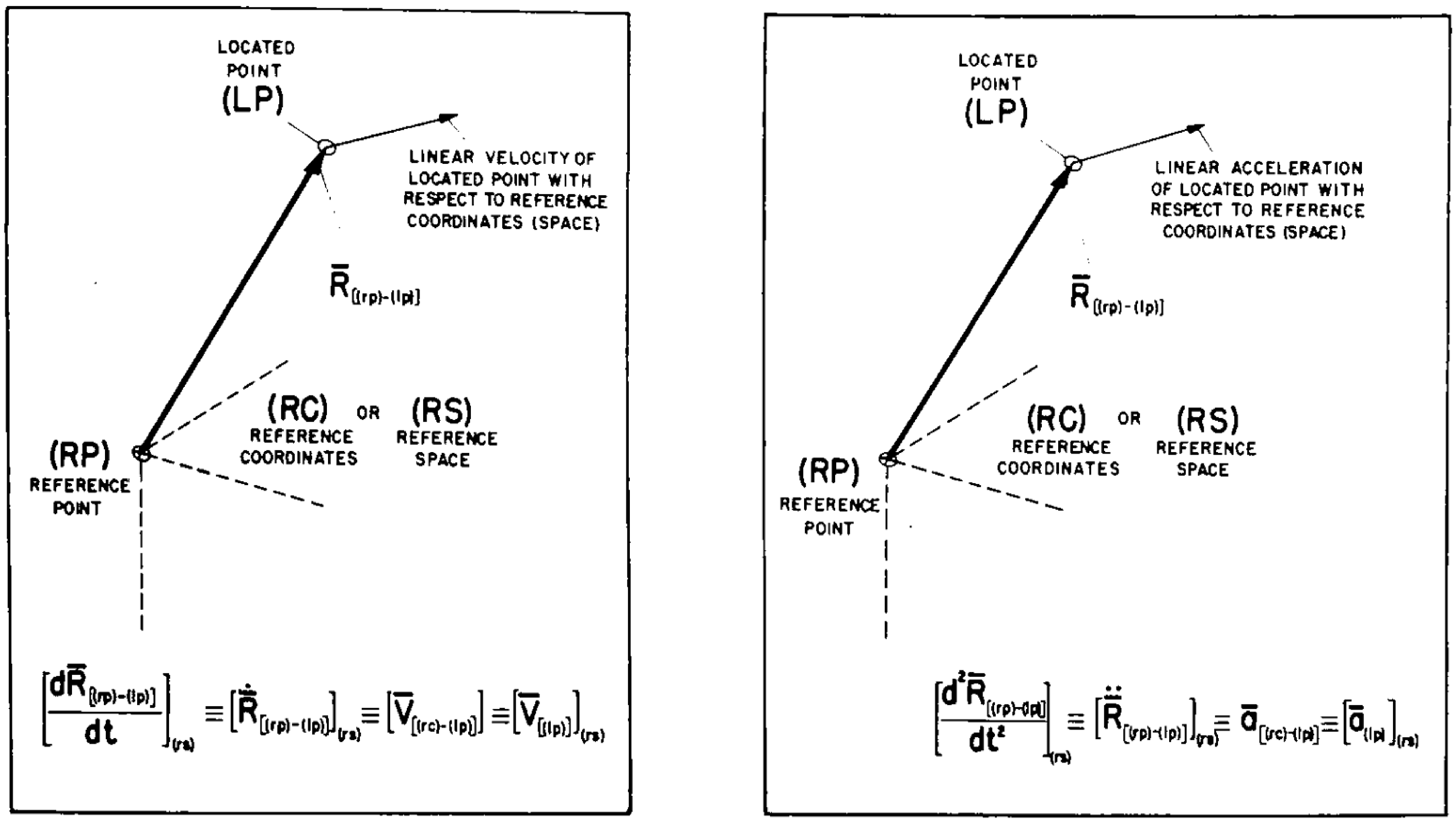
b) Atmospheric reference cylinder pressure as a typical example of a reference level compared quantity

Fig. 9-2. Symbols and definitions for reference level compared quantities.

CYLINDER PRESSURE AS AN EXAMPLE

9.6. Figure 9-2b illustrates the application of the concepts and symbol formation conventions to the case of pressure variations inside the cylinder of an engine or air pump. The reference pressure level is taken as the atmospheric pressure level (symbol p, with (atm), for atmospheric, as its subscript). The compared quantity is cylinder pressure (symbol p, with subscript (cyl)). As shown in Fig. 9-2b, the atmospheric reference cylinder pressure is equal to the difference between the cylinder pressure (the omission of a reference level in the name means that the reference level is zero) and atmospheric pressure.

9.7. The definitions and symbols of Figs. 9-1 and 9-2 are complete in that they convey all the essential information about the quantities represented. They are, in effect, self-defining symbols that to a considerable extent specify the quantities they represent. When symbol meanings are clear and a considerable amount of manipulation is to be carried out, it will generally be advantageous, for formal operations, to replace complete symbols with more convenient working variable symbols, or working symbols. However, it is generally worth while for any work that is done as part of an engineering project, rather than for educational training purposes, to be formulated and have its results stated in terms of self-defining symbols. When this rule is followed, information on the work will be easily available, not only immediately after analysis is completed when special symbols are fresh in the minds of the engineers involved, but also at any time thereafter when it becomes desirable to refer back to the recorded data.


LOCATED POINT
(LP)
LINEAR VELOCITY OF LOCATED POINT WITH RESPECT TO REFERENCE COORDINATES (SPACE)
$\bar{R}_{[(rp)-(lp)]}$
(RC) OR (RS)
REFERENCE COORDINATES
REFERENCE SPACE
(RP)
REFERENCE POINT
$\left[\frac{d\bar{R}_{[(rp)-(lp)]}}{dt}\right]_{(rs)} \equiv \left[\dot{\bar{R}}_{[(rp)-(lp)]}\right]_{(rs)} \equiv \left[\bar{V}_{[(rc)-(lp)]}\right] \equiv \left[\bar{V}_{[(lp)]}\right]_{(rs)}$
LOCATED POINT
(LP)
LINEAR ACCELERATION OF LOCATED POINT WITH RESPECT TO REFERENCE COORDINATES (SPACE)
$\bar{R}_{[(rp)-(lp)]}$
(RC) OR (RS)
REFERENCE COORDINATES
REFERENCE SPACE
(RP)
REFERENCE POINT
$\left[\frac{d^2\bar{R}_{[(rp)-(lp)]}}{dt^2}\right]_{(rs)} \equiv \left[\ddot{\bar{R}}_{[(rp)-(lp)]}\right]_{(rs)} \equiv \bar{a}_{[(rc)-(lp)]} \equiv \left[\bar{a}_{(lp)}\right]_{(rs)}$

a) Vector representation of linear velocity of located point with respect to reference coordinates (reference space)

b) Vector representation of linear acceleration of located point with respect to reference coordinates (reference space)

Fig. 9-3. Diagrams illustrating formation of symbols to represent the linear velocity and linear acceleration of a located point with respect to a reference space.

DIFFERENCES IN VECTOR QUANTITIES

9.8. Figure 9-3 illustrates the application of the conventions of Fig. 9-1 to the linear velocity and acceleration of a located point with respect to a set of reference coordinates that represent a reference space. The basic definition for the linear velocity of the located point is represented by a symbol written as the time derivative of the location vector, with its symbol formed by the conventions of Fig. 9-1. To show that this derivative represents a time rate of change with respect to the reference space, the conventional symbol is enclosed in square brackets and given the subscript (rs) to show that changes are considered with respect to the reference space. The derivative symbol may be simplified by replacing the conventional form, d/dt, with a dot over the vector symbol. A further simplification is possible if the letter v (for linear velocity) is used to represent the time rate of change of the vector R. The main symbol v is given a subscript bracket with (rc) in the comparison quantity position to show that the velocity is with respect to reference coordinates, and with (lp) in the compared quantity position to show that the velocity considered is that of the located point. The velocity may also be represented by the main symbol v, with (lp) as the direct subscript to show that the velocity is that of the located point. This symbol is then enclosed in square brackets and given the subscript (rs) to show that the located point velocity is with respect to the reference space.

9.9. Figure 9-3b illustrates the application of the conventions explained in Fig. 9-3a for linear velocity to the linear acceleration of a located point. This symbol is identical with that for velocity if it is considered that acceleration is a second derivative with respect to time instead of a first derivative with respect to time, and that the symbol a, for acceleration, replaces the symbol v, for velocity.

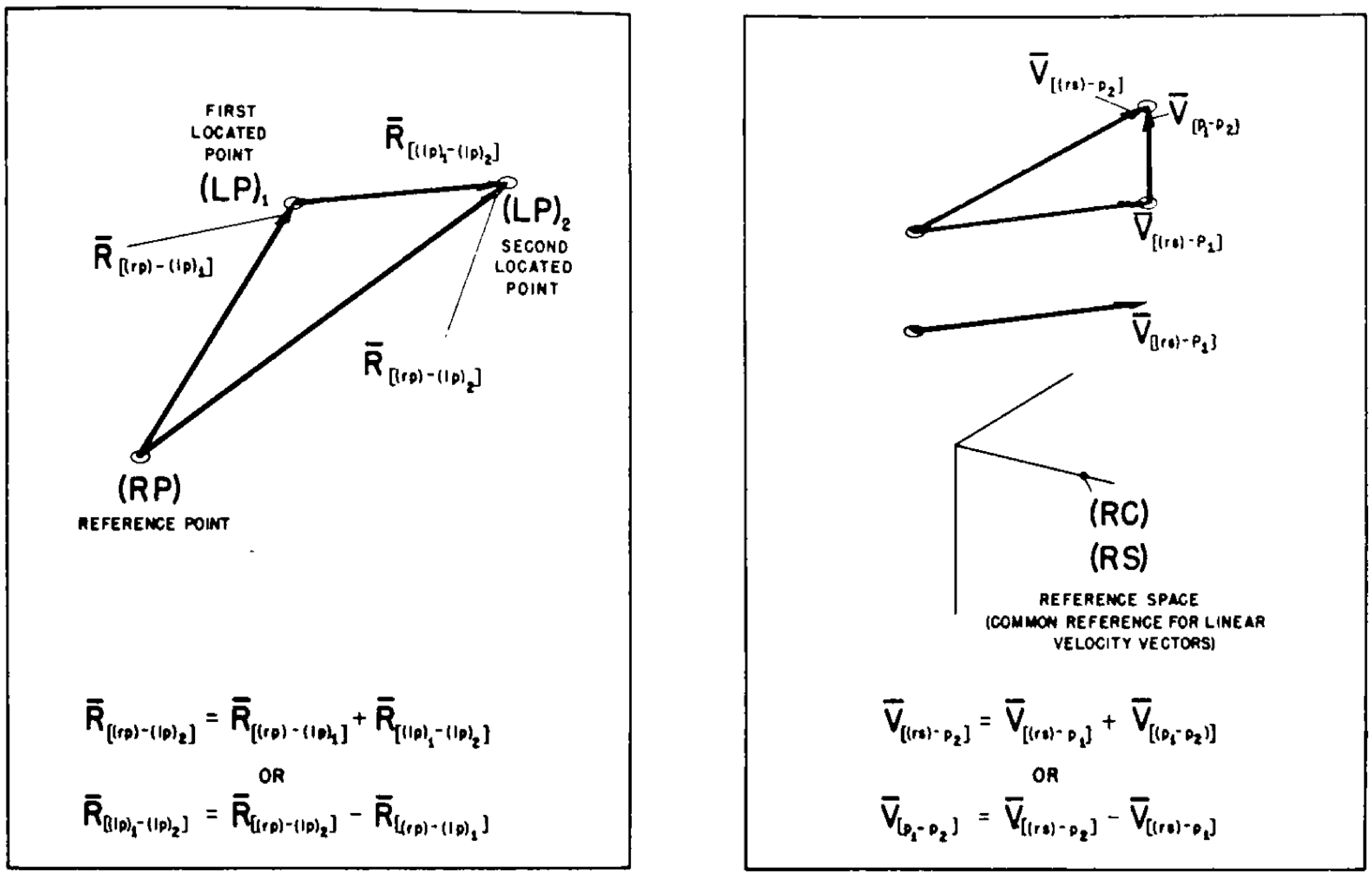
9.10. In practice, it is certainly worth while to take advantage of such typical simplifications as those suggested in Fig. 9-3 when the symbols are used in lengthy derivations and in the detailed work of analysis. On the other hand, when summarized results and principal equations are recorded for reference purposes, the reduction in memory effort that is made possible by use of complete symbols makes it worth while to employ self-defining symbols for a more generally useful expression of results.

SUBSCRIPT COMBINATION RULE FOR VECTORS

9.11. The notation illustrated in Figs. 9-1, 9-2, and 9-3 lends itself well to the representation of vector quantities. In addition to the conventions already discussed, it is useful to note that the addition or subtraction of location vectors using a common reference point may be performed by simple subscript manipulations. The subscript combination rule is illustrated in the diagram of Fig. 9-4a. This rule is

> *in a sum of location vectors, the symbol for the resultant vector may be formed by cancelling all subscripts in the reference point position with like subscripts in the located point position leaving a single symbol with the two remaining terminal subscripts written in the positions they occupy in the original vector symbols.*

For example, in the vector sum of Fig. 9-4a, the subscript $(lp)_1$ occurs in the located point subscript position for the first vector and the reference point subscript position for the second vector of the sum. The corresponding vector difference of Fig. 9-4a has a symbol in which the common reference point is dropped out and the remaining subscript point of the subtracted vector appears as the reference symbol for the difference vector.


FIRST LOCATED POINT
(LP)$_1$
$\bar{R}_{[(lp)_1-(lp)_2]}$
$\bar{R}_{[(rp)-(lp)_1]}$
(LP)$_2$
SECOND LOCATED POINT
$\bar{R}_{[(rp)-(lp)_2]}$
(RP)
REFERENCE POINT
$\bar{R}_{[(rp)-(lp)_2]} = \bar{R}_{[(rp)-(lp)_1]} + \bar{R}_{[(lp)_1-(lp)_2]}$
OR
$\bar{R}_{[(lp)_1-(lp)_2]} = \bar{R}_{[(rp)-(lp)_2]} - \bar{R}_{[(rp)-(lp)_1]}$
$\bar{V}_{[(rs)-p_2]}$
$\bar{V}_{(p_1-p_2)}$
$\bar{V}_{[(rs)-p_1]}$
$\bar{V}_{[(rs)-p_1]}$
(RC)
(RS)
REFERENCE SPACE
(COMMON REFERENCE FOR LINEAR VELOCITY VECTORS)
$\bar{V}_{[(rs)-p_2]} = \bar{V}_{[(rs)-p_1]} + \bar{V}_{[(p_1-p_2)]}$
OR
$\bar{V}_{[p_1-p_2]} = \bar{V}_{[(rs)-p_2]} - \bar{V}_{[(rs)-p_1]}$

a) Addition and subtraction of a location vector

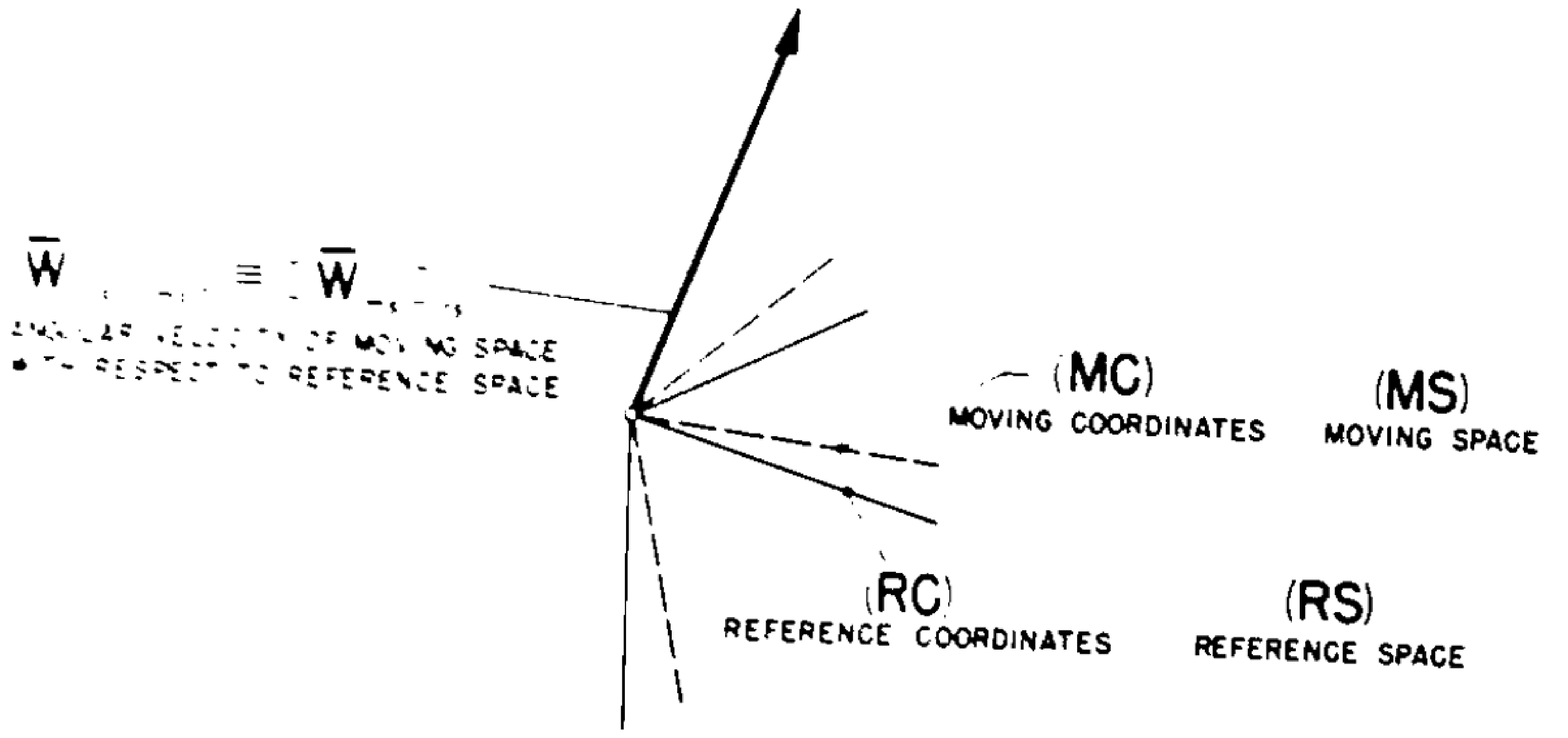
b) Addition and subtraction of a linear velocity vector

Fig. 9-4. Illustrations of the subscript combination rule for vector addition.

9.12. Figure 9-4b shows that the subscript combination rule holds for linear velocity vectors based on a common reference space in the same way that it applies for location vectors.

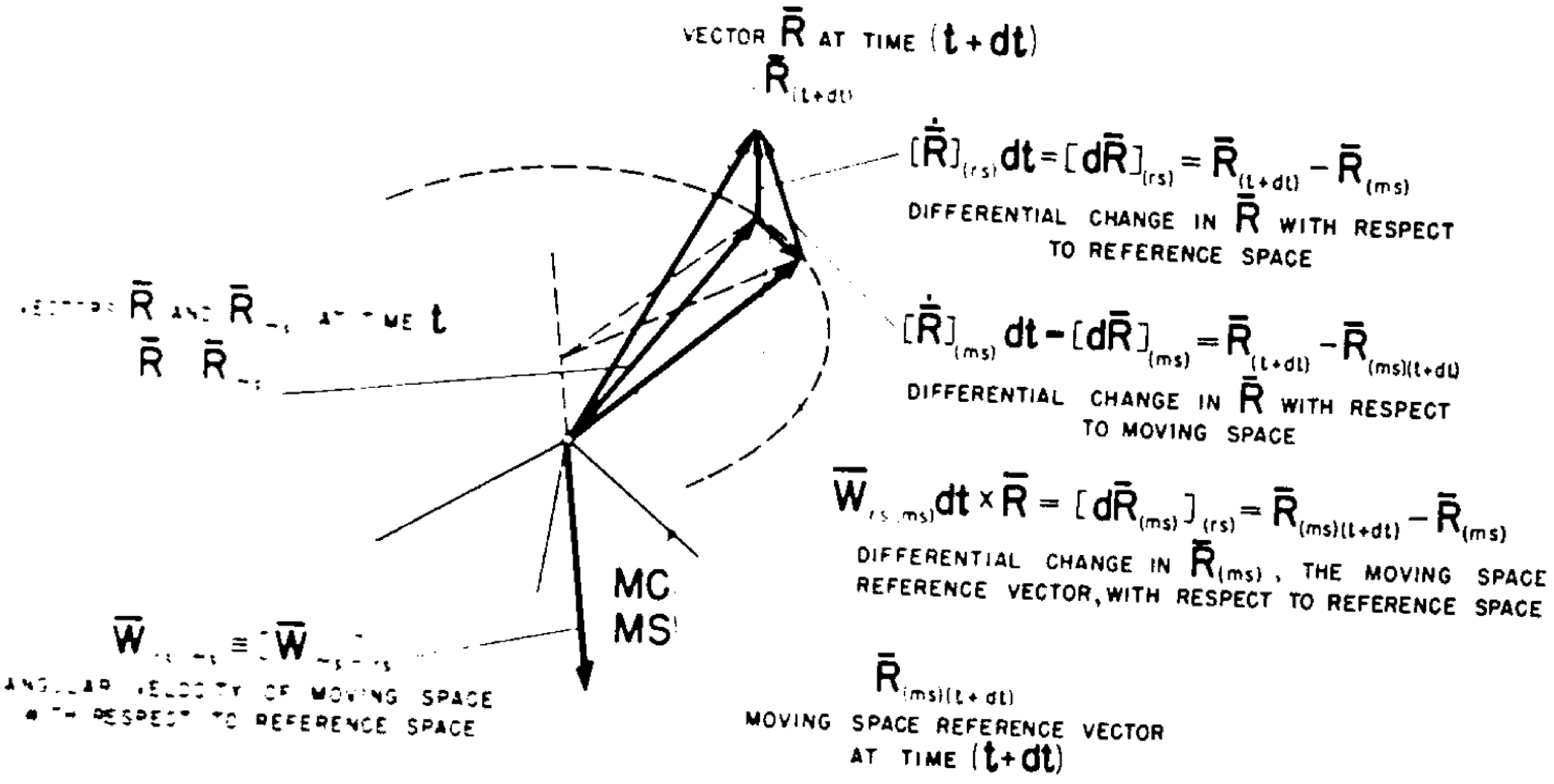
SYMBOLS FOR ANGULAR VELOCITY

9.13. Rules for the formation of symbols to represent angular velocities of a moving space (represented by a moving coordinate system) with respect to a reference space (represented by a reference coordinate system) are illustrated in the diagram of Fig. 9-5. The main symbol for angular velocity is W, since it is the nearest English equivalent to the Greek letter "omega," the symbol commonly used for angular velocity. Two conventions are convenient for forming the symbol. First, the symbol for the reference space may be placed in the



$\bar{W}$ ≡ $[\bar{W}]$
ANGULAR VELOCITY OF MOVING SPACE WITH RESPECT TO REFERENCE SPACE
(MC) MOVING COORDINATES
(MS) MOVING SPACE
(RC) REFERENCE COORDINATES
(RS) REFERENCE SPACE

a) Vector representation of angular velocity of moving space with respect to reference space



VECTOR $\bar{R}$ AT TIME $(t+dt)$
$\bar{R}_{(t+dt)}$
$[\dot{\bar{R}}]_{(rs)}dt = [d\bar{R}]_{(rs)} = \bar{R}_{(t+dt)} - \bar{R}_{(ms)}$
DIFFERENTIAL CHANGE IN $\bar{R}$ WITH RESPECT TO REFERENCE SPACE
$[\dot{\bar{R}}]_{(ms)}dt - [d\bar{R}]_{(ms)} = \bar{R}_{(t+dt)} - \bar{R}_{(ms)(t+dt)}$
DIFFERENTIAL CHANGE IN $\bar{R}$ WITH RESPECT TO MOVING SPACE
VECTORS $\bar{R}$ AND $\bar{R}_{(ms)}$ AT TIME t
$\bar{R}$ $\bar{R}_{(ms)}$
$\bar{W}_{(rs)(ms)}dt \times \bar{R} = [d\bar{R}_{(ms)}]_{(rs)} = \bar{R}_{(ms)(t+dt)} - \bar{R}_{(ms)}$
DIFFERENTIAL CHANGE IN $\bar{R}_{(ms)}$, THE MOVING SPACE REFERENCE VECTOR, WITH RESPECT TO REFERENCE SPACE
MC
MS
$\bar{W}$ ≡ $[\bar{W}]$
ANGULAR VELOCITY OF MOVING SPACE WITH RESPECT TO REFERENCE SPACE
$\bar{R}_{(ms)(t+dt)}$
MOVING SPACE REFERENCE VECTOR AT TIME $(t+dt)$

THE EQUATION EXPRESSING THE LAW OF CORIOLIS IS:

$$[\dot{\bar{R}}]_{(rs)} = [\dot{\bar{R}}]_{(ms)} + \bar{W}_{(rs)(ms)} \times \bar{R}$$

$$\left[\begin{array}{c}\text{TIME RATE OF CHANGE OF } \bar{R} \\ \text{WITH RESPECT TO REFERENCE SPACE}\end{array}\right] = \left[\begin{array}{c}\text{TIME RATE OF CHANGE OF } \bar{R} \\ \text{WITH RESPECT TO MOVING SPACE}\end{array}\right] + \left[\begin{array}{c}\text{CROSS PRODUCT OF ANGULAR} \\ \text{VELOCITY OF MOVING SPACE} \\ \text{WITH RESPECT TO REFERENCE} \\ \text{SPACE WITH VECTOR } \bar{R}\end{array}\right]$$

b) Vector representation of law of Coriolis

Fig. 9-5. Diagrams illustrating formation of symbols for angular velocity of a moving space with respect to a reference space and for the time rate of change of a vector with respect to reference space in terms of the time rate of change of the vector with respect to a moving space and the angular velocity of the moving space with respect to the reference space.

subscript position nearest the main symbol with the moving-space symbol in the next following position. In the second convention, the moving-space symbol is directly adjacent to the main symbol to indicate that it is the space that has the angular velocity, and the reference-space symbol is placed as a subscript to square brackets enclosing the main symbol and its identifying subscript.

SYMBOLS ASSOCIATED WITH ANGULAR VELOCITIES AMONG COORDINATE SYSTEMS

9.14. Figure 9-5 illustrates the formation of symbols associated with the Law of Coriolis,(12) which gives the expression for the time rate of change of a vector $\bar{R}$ with respect to a reference space in terms of the time rate of change of $\bar{R}$ with respect to a moving space and the cross product of the angular velocity of the moving space with respect to the reference space and $\bar{R}$. The superscript dot notation is used to indicate differentiation with respect to time, and a subscript (rs) (meaning reference space) is placed outside the square brackets enclosing the symbol to specify that the time derivative is taken with respect to reference space. The symbol for the time rate of change of $\bar{R}$ with respect to the moving space is similar to that for the reference space time derivative except that the subscript (rs) is replaced by the subscript (ms).

9.15. The main symbol W, with the subscript (rs)(ms), represents the angular velocity of the moving space (ms) with respect to the reference space (rs). The complete symbol for this angular velocity omits the square bracket and dash between symbols inside the bracket that are used in Fig. 9-5a. This omission is made to illustrate the type of notational simplifications that are possible. If, for a certain situation, the angular velocities of a number of coordinate systems are involved, and it is understood that all these angular velocities are to be taken with respect to the same reference space, the reference symbol (rs) may be omitted from the subscript.

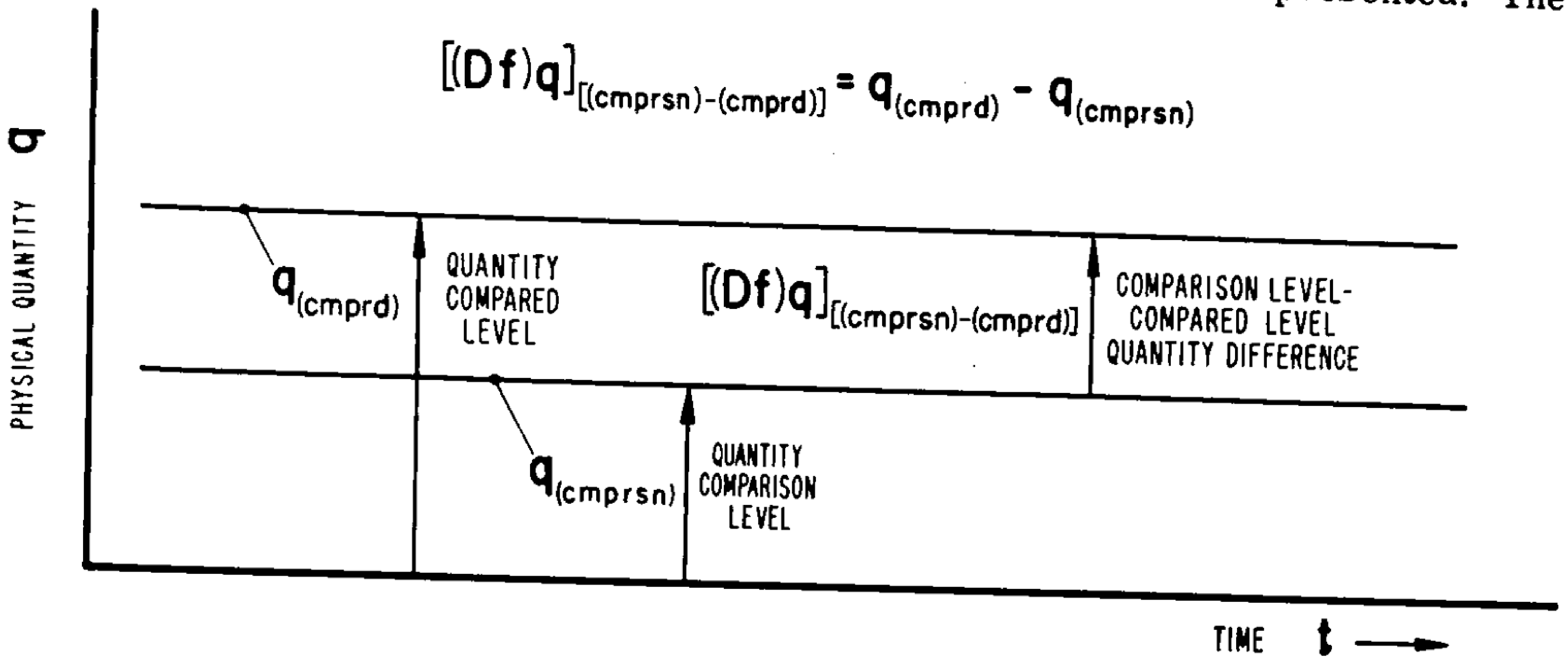
GENERAL CONCEPTS OF LEVEL COMPARISON

9.16. The reference level — compared level quantity difference concepts of the preceding section apply without expressed or implied restrictions on the magnitudes of reference level — compared level differences. In practice, differences of this kind between quantity levels of substantially equal magnitudes are especially important. For this reason, it is useful to set up definitions and symbols for a group of level comparison concepts. These concepts are defined and illustrated with examples in this chapter.

9.17. Any system of definitions and symbols developed for level comparison purposes must include provisions for two basic kinds of relationships. The first type of relationship is the algebraic difference between two levels of the same physical quantity. The second relationship type is the ratio between two levels of the same quantity. These two types of comparison measures are discussed below.

QUANTITY DIFFERENCE

9.18. Figure 9-6 illustrates the concept of a simple difference between typical quantity levels under static conditions. In order to systematically establish the algebraic sign for a quantity difference, it is necessary to distinguish between the two quantity levels involved. By definition, the quantity comparison level is the level taken as the standard of reference. This comparison level is subtracted from the quantity compared level to give the comparison level – compared level quantity difference. The symbol for this difference is (Df),* enclosed in parentheses to differentiate the symbol from a factor of a product. The subscript enclosed in square brackets has (cmprsn) as its first symbol, to the left, and (cmprd) as its second symbol. These two symbols are intended to suggest unambiguously "comparison" and "compared" by omitting all letters except those that contribute to the characteristic shape of the word to be represented. The


$[(Df)q]_{[(cmprsn)-(cmprd)]} = q_{(cmprd)} - q_{(cmprsn)}$
PHYSICAL QUANTITY q
$q_{(cmprd)}$
QUANTITY COMPARED LEVEL
$[(Df)q]_{[(cmprsn)-(cmprd)]}$
COMPARISON LEVEL-COMPARED LEVEL QUANTITY DIFFERENCE
$q_{(cmprsn)}$
QUANTITY COMPARISON LEVEL
TIME t

In the definition of Quantity Difference both Comparison and Compared Quantity Levels are typical levels with no special properties. Both Levels exist at the same time.

Fig. 9-6. Definition and symbols for quantity difference.

complete subscript illustrates the general convention for expressing a comparison in symbol form with the subscript representing the comparison quantity placed within square brackets in the position nearest the main symbol (in this case (Df)), and the symbol representing the compared quantity placed within the square brackets to the right of the comparison quantity symbol. The various conventions and symbols are illustrated in Fig. 9-6, which also shows that

> *when quantity levels are represented by horizontal lines, an arrow representing the quantity difference is drawn from the comparison level to the compared level.*

QUANTITY CHANGE

9.19. In Fig. 9-6, quantity difference is defined in terms of two simultaneously existing levels. Another situation exists when the levels considered do not exist at the same time, but are associated with a change of the physical quantity involved.

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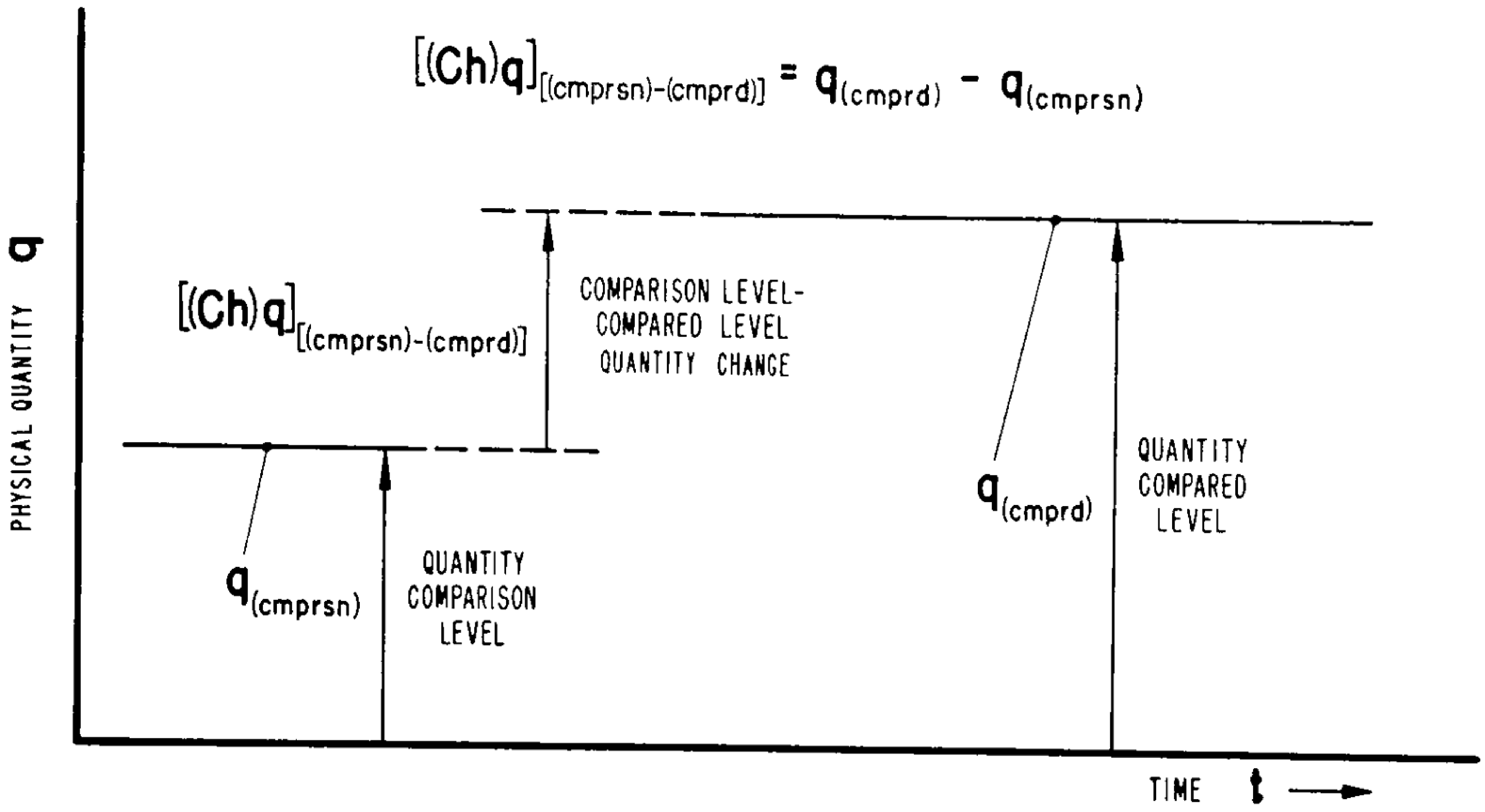
* It is recognized that the Greek letter Δ is commonly used to represent differences. In this book, Δ is used with the same meaning as a working variable rather than as part of a self-defining symbol.

A case of this kind is represented in Fig. 9-7, where the comparison quantity is taken as one constant level and the compared quantity is the constant level reached after a change has occurred. The main symbol for this change is (Ch)q, where the letters Ch represent "change" and are enclosed in parentheses to show that the change occurs in the quantity that follows – in this case, the quantity q. The subscript bracket containing the two symbol forms (cmprsn) and (cmprd) provides a means for identifying the levels between which the change occurs. As in the case of the quantity difference, defined in Fig. 9-6, quantity change is equal to the compared quantity level minus the comparison quantity level.


$[(Ch)q]_{[(cmprsn)-(cmprd)]} = q_{(cmprd)} - q_{(cmprsn)}$
$[(Ch)q]_{[(cmprsn)-(cmprd)]}$
COMPARISON LEVEL-COMPARED LEVEL QUANTITY CHANGE
$q_{(cmprd)}$
QUANTITY COMPARED LEVEL
$q_{(cmprsn)}$
QUANTITY COMPARISON LEVEL
PHYSICAL QUANTITY q
TIME t

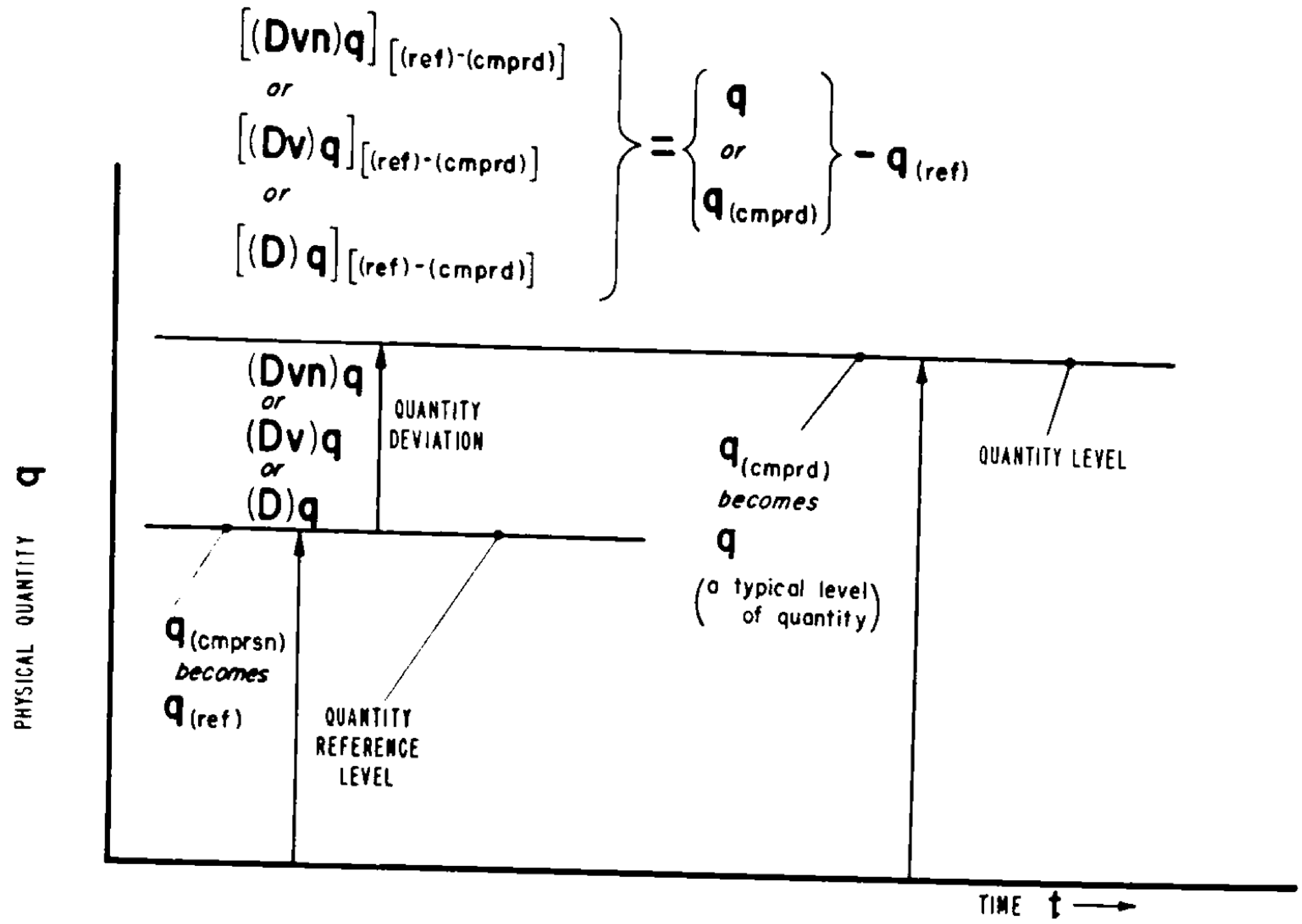
In the definition of Quantity Change the Comparison Level and the Compared Level do not exist at the same time. The Compared Level exists after the Comparison Level has ceased to exist. With this exception both Comparison and Compared Levels are typical levels of the Quantity q

Fig. 9-7. Definition and symbols for quantity change.

DEVIATION

9.20. The quantity difference and change definitions are based on typical levels of the quantity involved. In practice, it is often useful to take a special level as the comparison level and to identify it as the reference level. Differences associated with a reference level are called quantity deviations and are assigned the symbol (Dvn)q. As shown in Fig. 9-8, the deviation of a given quantity level q is equal to the given level minus the reference level. When no misunderstanding is possible, the symbol (Dvn), which represents deviation, may be replaced by either of the simpler forms (Dv) or (D). The latter symbol is the form commonly used to represent deviations.

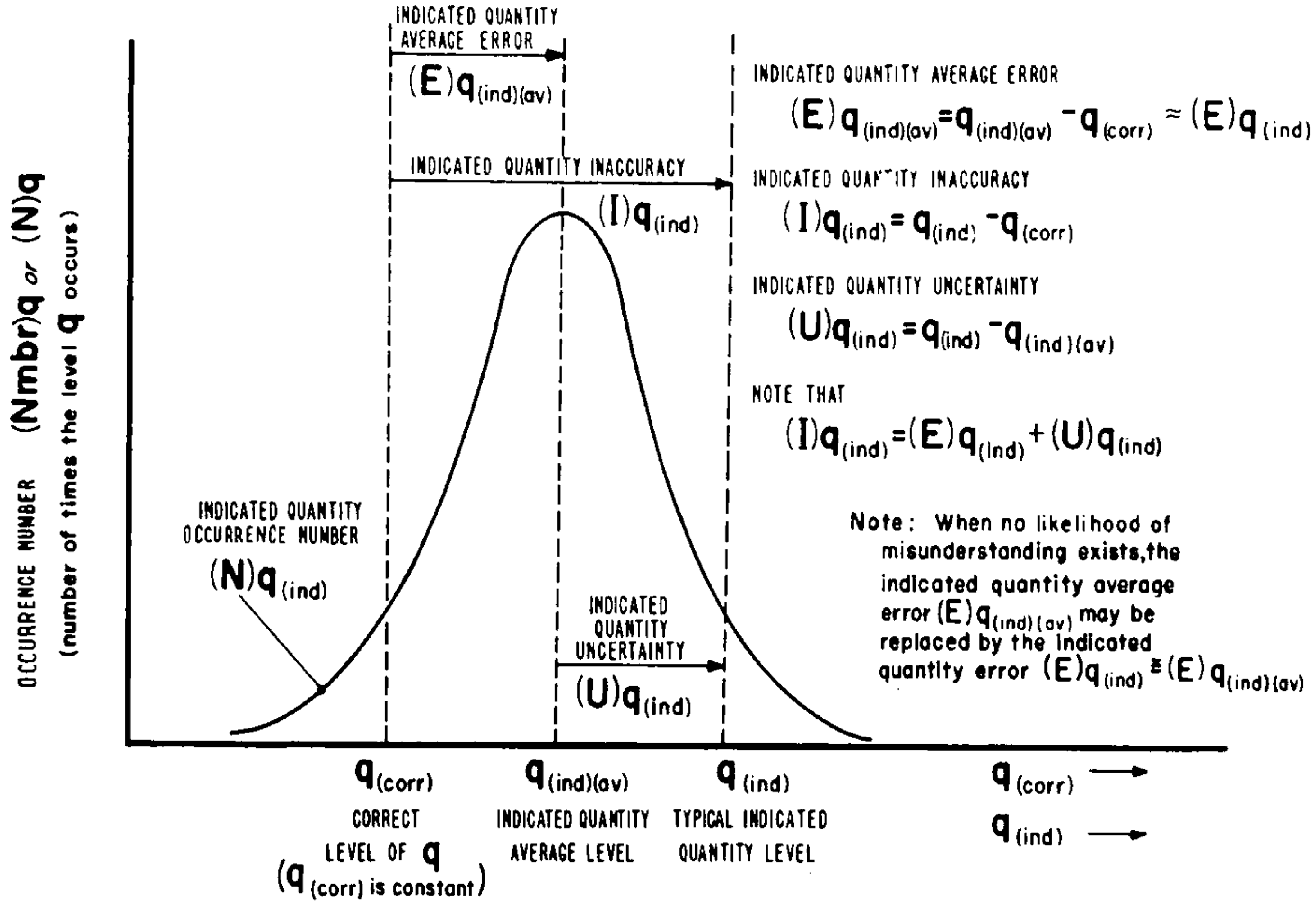
9.21. Differences have special significance when the comparison quantity is the true level, or the correct level, while the compared quantity is a level that would be identical with the true level if the operating system with which it is associated were ideal. For example, the comparison quantity might be the true pressure and the compared quantity might be taken as the indication generated by a pressure gage of the type shown in Fig. 2-2. Various concepts are associated with differences of this type, depending upon the level used for comparison purposes. When conditions are static, so that repeated indications of a fixed pressure level are possible, three useful difference concepts are possible. These three concepts are illustrated in Fig. 9-9 by means of a curve showing the number of occurrences of various indicated quantity levels (the symbol $(N)q_{(ind)}$ represents the number of times the indicated quantity level $q_{(ind)}$ occurs in a series of observations) when the correct level of the quantity is constant. An important level associated with this distribution curve is the average of the indicated quantity, which must be determined for a group of indicated quantity levels by a


$[(Dvn)q]_{[(ref)-(cmprd)]}$ or $[(Dv)q]_{[(ref)-(cmprd)]}$ or $[(D)q]_{[(ref)-(cmprd)]}$ = {q or $q_{(cmprd)}$} − $q_{(ref)}$
(Dvn)q
or
(Dv)q
or
(D)q
QUANTITY DEVIATION
$q_{(cmprd)}$ becomes q (a typical level of quantity)
QUANTITY LEVEL
$q_{(cmprsn)}$ becomes $q_{(ref)}$
QUANTITY REFERENCE LEVEL
PHYSICAL QUANTITY q
TIME t →

In the definition of Quantity Deviation the Reference Level and the Compared Level may or may not exist simultaneously. The Reference Level has some special property, either assigned arbitrarily or because of some natural quality

Fig. 9-8. Definition and symbols for quantity deviation.

statistical procedure.* As shown in Fig. 9-9, when the correct indicated quantity is taken as the comparison level and the compared quantity is the indicated quantity average, the corresponding difference is the *indicated quantity error*. When the comparison quantity is the correct quantity and the compared quantity is a single typical quantity, the difference is a particular value of the *indicated quantity inaccuracy*. The *indicated quantity uncertainty* for a particular case is the *quantity difference* when the comparison quantity is the indicated quantity average and the compared quantity is a single indicated quantity level. These definitions mean that for any given single observation of the indicated quantity, *the inaccuracy is equal to the sum of the error and the uncertainty*.


OCCURRENCE NUMBER (Nmbr)q or (N)q
(number of times the level q occurs)
INDICATED QUANTITY AVERAGE ERROR
$(E)q_{(ind)(av)}$
INDICATED QUANTITY INACCURACY
$(I)q_{(ind)}$
INDICATED QUANTITY OCCURRENCE NUMBER
$(N)q_{(ind)}$
INDICATED QUANTITY UNCERTAINTY
$(U)q_{(ind)}$
INDICATED QUANTITY AVERAGE ERROR
$(E)q_{(ind)(av)} = q_{(ind)(av)} - q_{(corr)} \approx (E)q_{(ind)}$
INDICATED QUANTITY INACCURACY
$(I)q_{(ind)} = q_{(ind)} - q_{(corr)}$
INDICATED QUANTITY UNCERTAINTY
$(U)q_{(ind)} = q_{(ind)} - q_{(ind)(av)}$
NOTE THAT
$(I)q_{(ind)} = (E)q_{(ind)} + (U)q_{(ind)}$
$q_{(corr)}$
CORRECT LEVEL OF q
($q_{(corr)}$ is constant)
$q_{(ind)(av)}$
INDICATED QUANTITY AVERAGE LEVEL
$q_{(ind)}$
TYPICAL INDICATED QUANTITY LEVEL
$q_{(corr)}$ →
$q_{(ind)}$ →

Note: When no likelihood of misunderstanding exists, the indicated quantity average error $(E)q_{(ind)(av)}$ may be replaced by the indicated quantity error $(E)q_{(ind)} \cong (E)q_{(ind)(av)}$

Fig. 9-9. Definitions and symbols for inaccuracy, error, and uncertainty for situations where repeated trials are possible under static conditions.

9.22. In practical applications of the definitions given in Fig. 9-9, inaccuracy represents the total difference by which a given single indication misses being correct. The inaccuracy may in general be resolved into two parts: one of these, the error, is systematic; the other, the uncertainty, is erratic; so that in any given instance it is not exactly predictable but must be treated by the methods of statistical analysis, which are discussed in Chapter 11. The concepts of inaccuracy, uncertainty, and error may be applied to the output of any operating system if the quasi-static output is taken as the "correct" level and the output itself is considered as the "indicated quantity."

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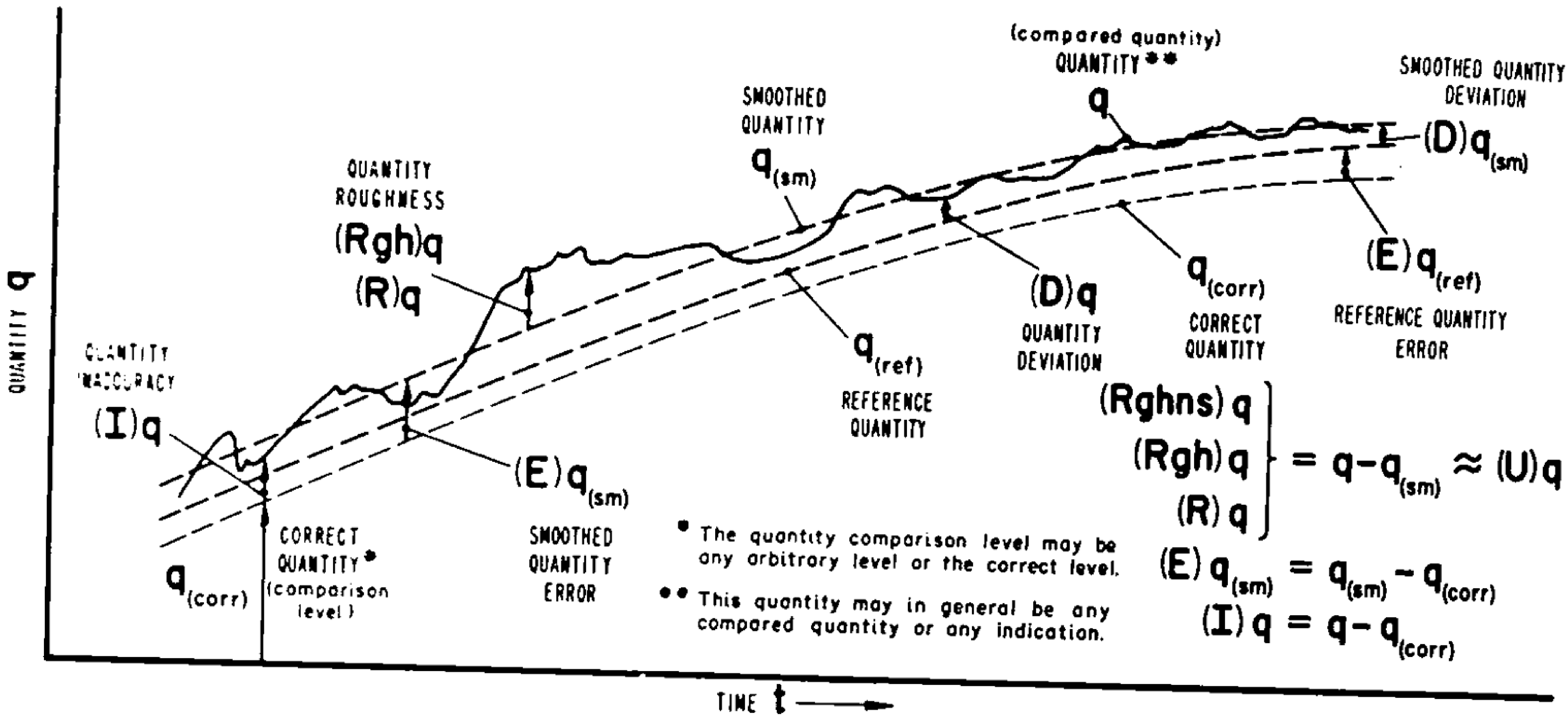
* The statistical method for static problems is discussed in Chapter 11.

DIFFERENCE CONCEPTS ASSOCIATED WITH QUANTITY LEVELS VARYING WITH TIME

9.23. The situations of instrument engineering seldom permit a series of output observations corresponding to a fixed level of the input quantity. This means that the definitions summarized in Fig. 9-9 must be supplemented by concepts that may be used when the compared quantity and the comparison quantity are both varying so rapidly with time that no repetition of observations is possible. In practice, continuous variations of the compared quantity may be known from records, and a reference level established by an independent means may be available. Because repeated observations of a given level of the compared quantity are not possible, the conventional statistical technique of averaging is not available. However, it is always possible to use the concept of smoothing already defined in Chapter 6 to establish a smoothed compared quantity, which corresponds to the average of a series of repeated observations for static situations. Figure 9-10 illustrates the relationships among the compared quantity taken as the "quantity," or "indicated quantity," the reference quantity, and the comparison quantity considered as the correct quantity. By definition, on the figure, when the smoothed level is taken as the comparison quantity and the indicated quantity is taken as the compared quantity, the difference is the quantity roughness. Under this definition, the roughness component for a given point of the compared quantity curve corresponds to the uncertainty of a single observation when repeated static observations are possible at a single input level.



* The quantity comparison level may be any arbitrary level or the correct level.

** This quantity may in general be any compared quantity or any indication.

$$\left.\begin{array}{r}(Rghns)q\\(Rgh)q\\(R)q\end{array}\right\} = q-q_{(sm)} \approx (U)q$$

$$(E)q_{(sm)} = q_{(sm)} - q_{(corr)}$$

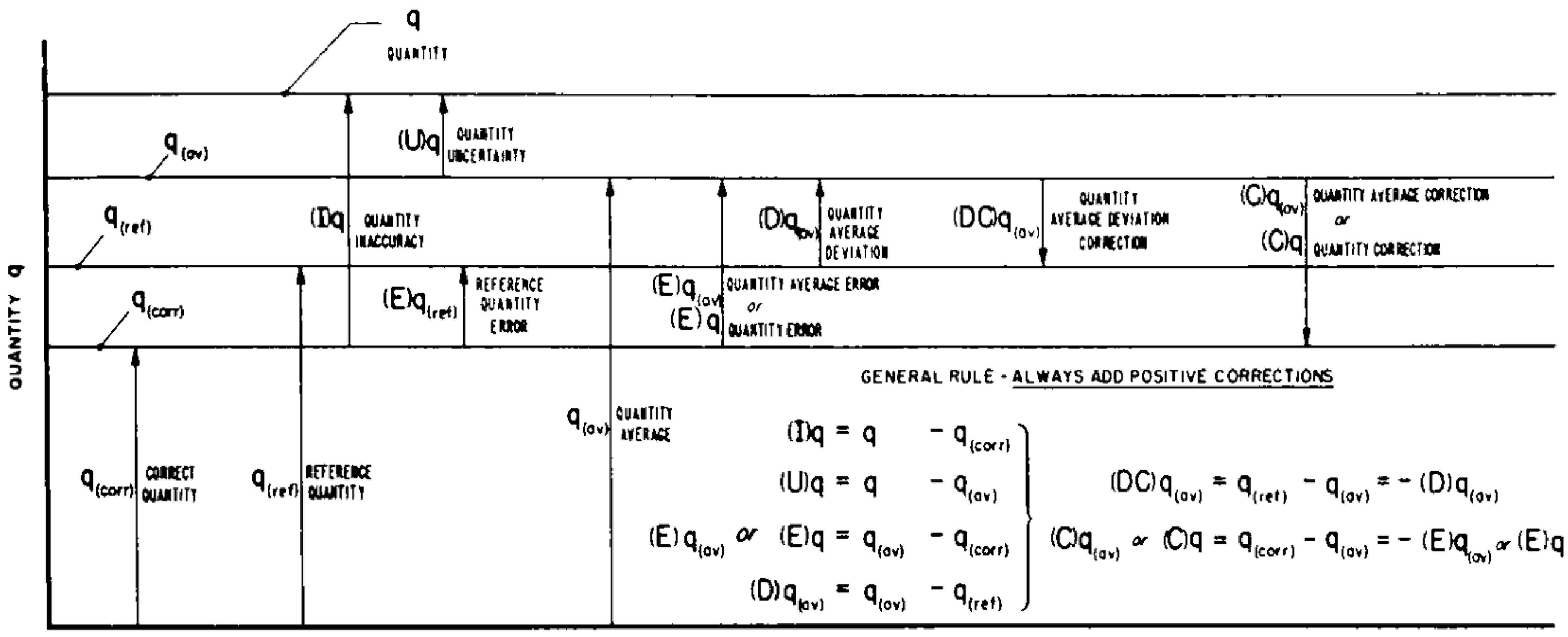
$$(I)q = q - q_{(corr)}$$

Fig. 9-10. Graphical definitions of inaccuracy, roughness, and error for a quantity varying with time.

9.24. In Fig. 9-10, the error of the smoothed quantity is the difference that appears when the correct quantity is taken as the comparison quantity and the smoothed quantity is taken as the compared quantity. Similarly, the deviation of the smoothed quantity is the difference based on the reference quantity as the comparison quantity and the smoothed indicated quantity as the compared quantity. The error of the reference quantity is the difference formed by taking the correct quantity as the comparison quantity and the reference quantity as the compared quantity. These definitions and certain others with obvious meanings are illustrated in Fig. 9-10.

CORRECTIONS

9.25. For the practical purposes of instrument engineering, information on deviations and errors is important only because it may be used as the basis for generating corrections. By definition, a correction for a quantity average is the amount by which the average level would have to be changed in order to bring it into agreement with the correct level. Similarly, a deviation correction is the amount by which a quantity average would have to be changed in order to make it identical with the reference quantity. Corresponding definitions hold for changing conditions in which smoothed quantities are substituted for the average quantities of static conditions. The correction definitions and symbols are summarized in Fig. 9-11 for both static and dynamic conditions. A consistent rule for all corrections based on the definitions of Fig. 9-11 is


q
QUANTITY
$q_{(av)}$
(U)q QUANTITY UNCERTAINTY
$q_{(ref)}$
(I)q QUANTITY INACCURACY
$(D)q_{(av)}$ QUANTITY AVERAGE DEVIATION
$(DC)q_{(av)}$ QUANTITY AVERAGE DEVIATION CORRECTION
$(C)q_{(av)}$ or (C)q QUANTITY AVERAGE CORRECTION or QUANTITY CORRECTION
$q_{(corr)}$
$(E)q_{(ref)}$ REFERENCE QUANTITY ERROR
$(E)q_{(av)}$ or (E)q QUANTITY AVERAGE ERROR or QUANTITY ERROR
QUANTITY q
GENERAL RULE - ALWAYS ADD POSITIVE CORRECTIONS
$q_{(av)}$ QUANTITY AVERAGE
$q_{(corr)}$ CORRECT QUANTITY
$q_{(ref)}$ REFERENCE QUANTITY

$$\left.\begin{aligned} (I)q &= q - q_{(corr)} \\ (U)q &= q - q_{(av)} \\ (E)q_{(av)} \text{ or } (E)q &= q_{(av)} - q_{(corr)} \\ (D)q_{(av)} &= q_{(av)} - q_{(ref)} \end{aligned}\right\} \quad \begin{aligned} (DC)q_{(av)} &= q_{(ref)} - q_{(av)} = -(D)q_{(av)} \\ (C)q_{(av)} \text{ or } (C)q &= q_{(corr)} - q_{(av)} = -(E)q_{(av)} \text{ or } (E)q \end{aligned}$$

When quotities are varying with time, the Average Quantity Level has no application — it must then be replaced by the smoothed quantity, i.e, $q_{(av)} \longrightarrow q_{(sm)}$.
THEN:

$$(E)q_{(sm)} = q_{(sm)} - q_{(corr)} \qquad (C)q_{(sm)} = q_{(corr)} - q_{(sm)} = -(E)q_{(sm)}$$

$$(D)q_{(sm)} = q_{(sm)} - q_{(ref)} \qquad (DC)q_{(sm)} = q_{(ref)} - q_{(sm)} = -(D)q_{(sm)}$$

Fig. 9-11. Summary of definitions and symbols for level comparison measures; definitions and symbols for corrections.

always add positive corrections.

ILLUSTRATIVE EXAMPLE

9.26. Figure 9-12 illustrates an application of the concepts and conventions described in Figs. 9-6 to 9-11. The diagram of Fig. 9-12a represents an indicator for making records of pressure variations occurring within the cylinder of an engine. The full-line curve of Fig. 9-12b shows the indicated cylinder pressure as taken with an instrument like that of Fig. 9-12a. The dashed-line curve represents a reference cylinder pressure variation that may be established either by an indicator more accurate than the equipment used for taking the full-line curve, or by theoretical considerations. The dotted-line curve of Fig. 9-12b represents the correct cylinder pressure variation that might, for example, be based upon records from a special, highly accurate, indicator. The indicated cylinder pressure correction, the cylinder pressure deviation correction, and the reference cylinder pressure correction are shown graphically in Fig. 9-12b. Figure 9-12c includes curves of the three corrections from Fig. 9-12b, and also gives the error curves corresponding to the correction curves. Because of the defining convention that

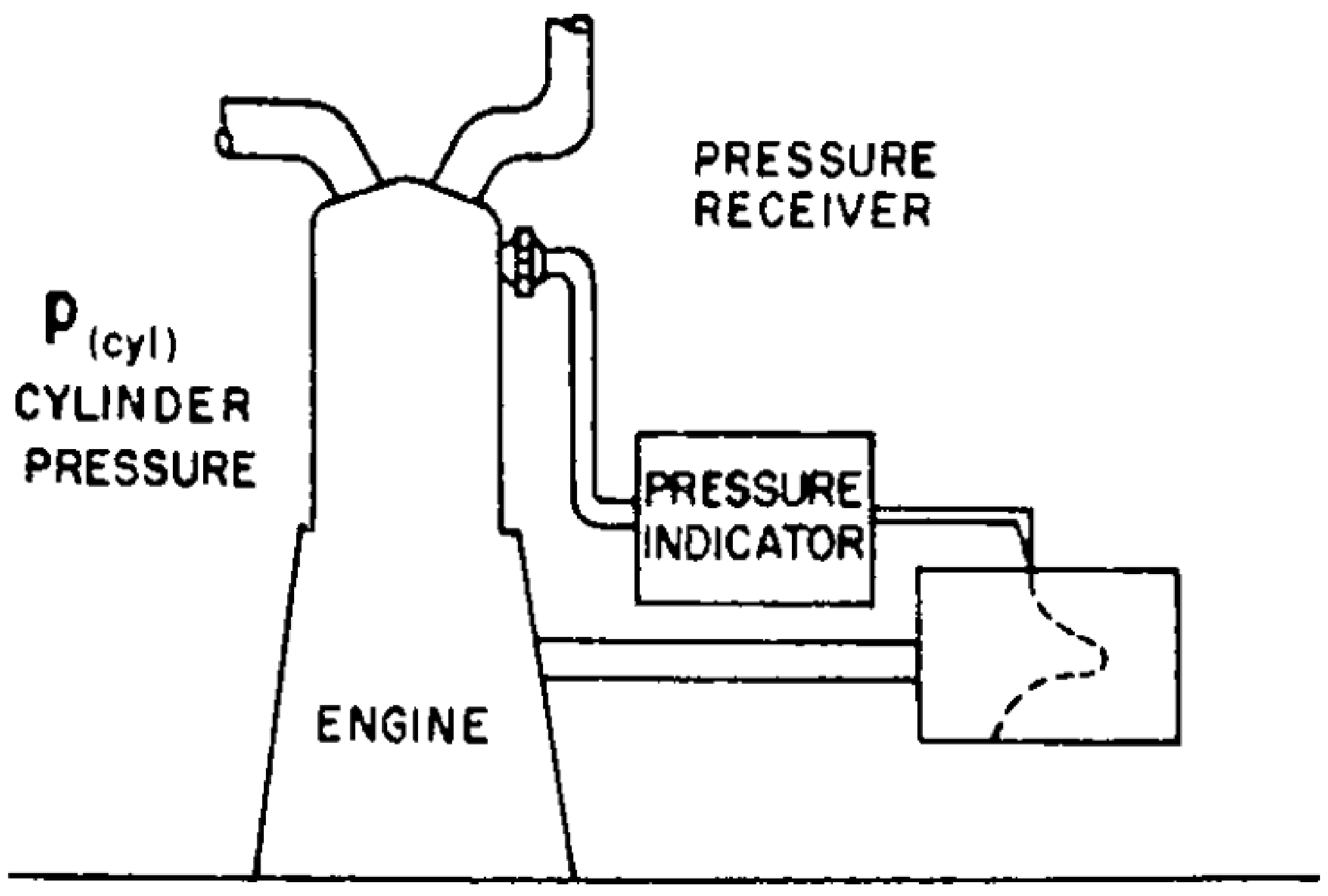
> *corrections are always equal in magnitude and opposite in sign to the corresponding errors*

the error curves are mirror images with respect to the zero correction axis of the correction curves.

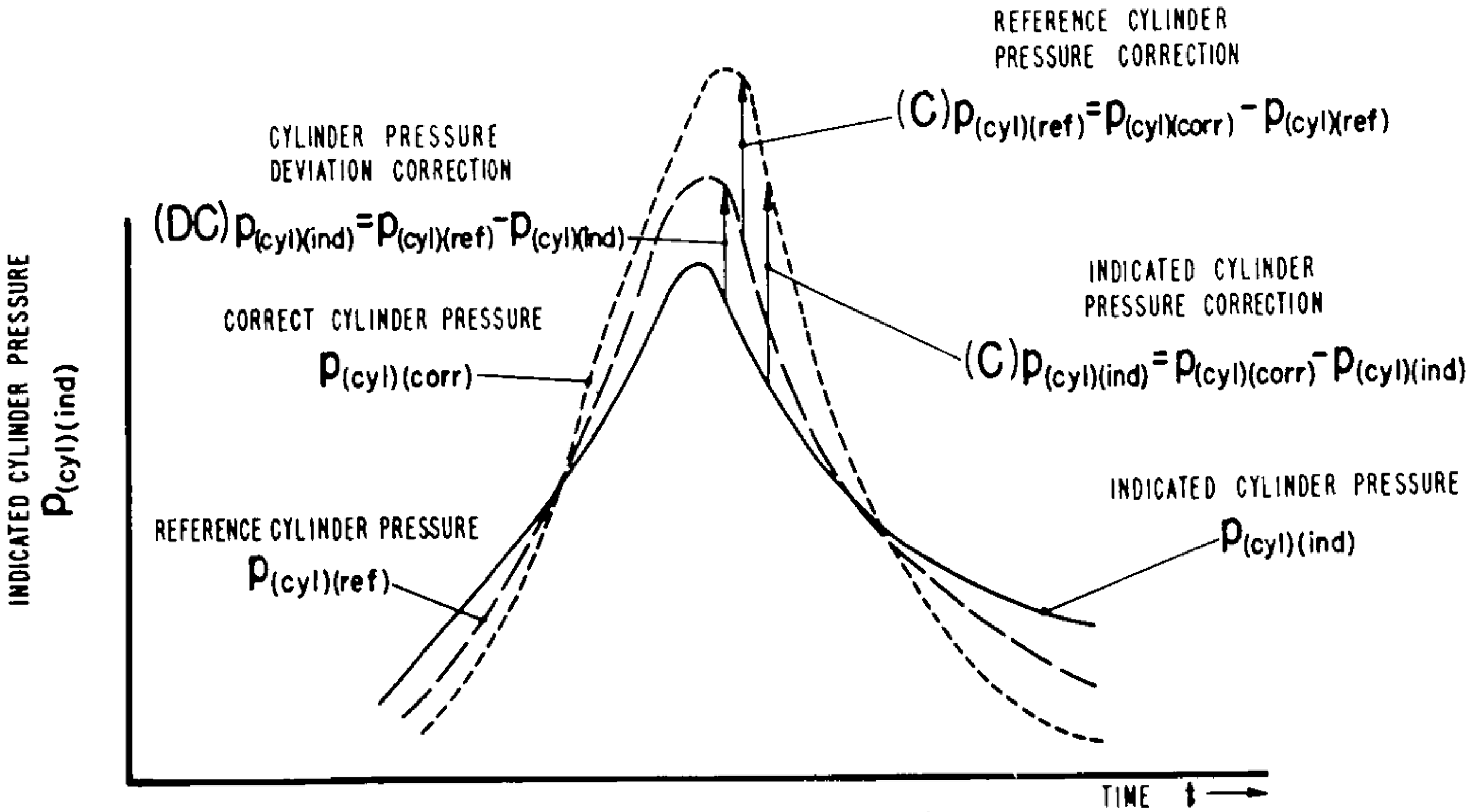
9.27. It is easily recognized that the definitions of deviations and errors given in this chapter are merely generalizations of the concepts already discussed in Chapter 5 on static performance. In general, presentations of deviation and error data are conveniently made by means of the linear, logarithmic, or semi-logarithmic plots described and illustrated by examples in Chapter 5. The extension of deviation and error definitions and methods of representation from static to dynamic conditions is a simple step that presents no special features. Many examples in which the concepts of deviation, error, and uncertainty are applied will appear later in the book.

TOLERANCE

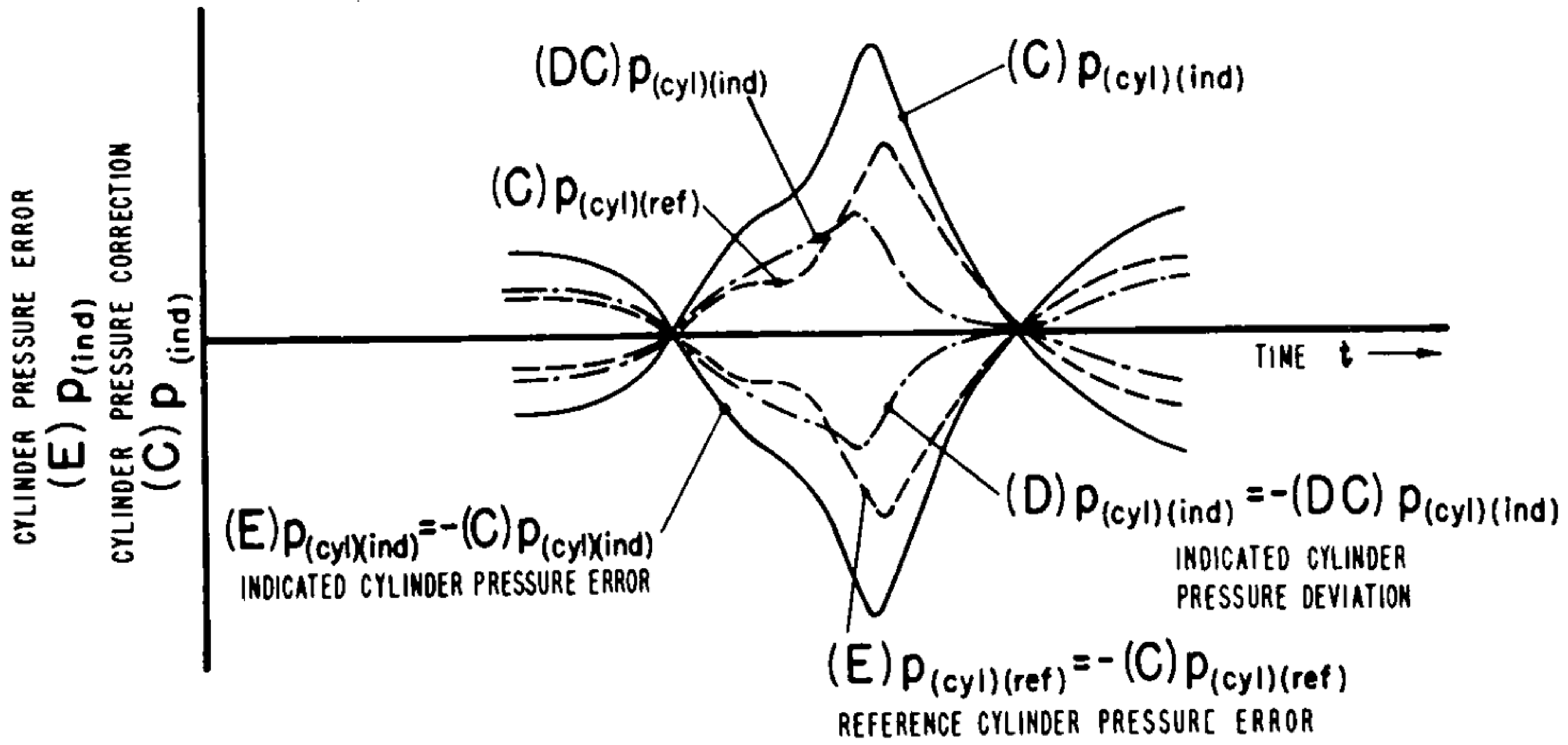
9.28. The use of tolerances for discussing operating system performance is necessary to properly describe and specify limits of satisfactory system performance. A set of definitions and symbols suitable for this purpose is given in Fig. 9-13, wherein a performance-describing quantity is represented by the symbol q and plotted as the ordinate. The operating level ratio, which is the ratio of the operating range quantity level to some chosen reference operating range quantity level, is plotted on the abscissa scale. For this purpose the operating range quantity is selected for its relationship to the factors controlling tolerance effects. Some level of the performance-describing quantity, q, for which the



a) Engine and indicator for making indicated pressure record



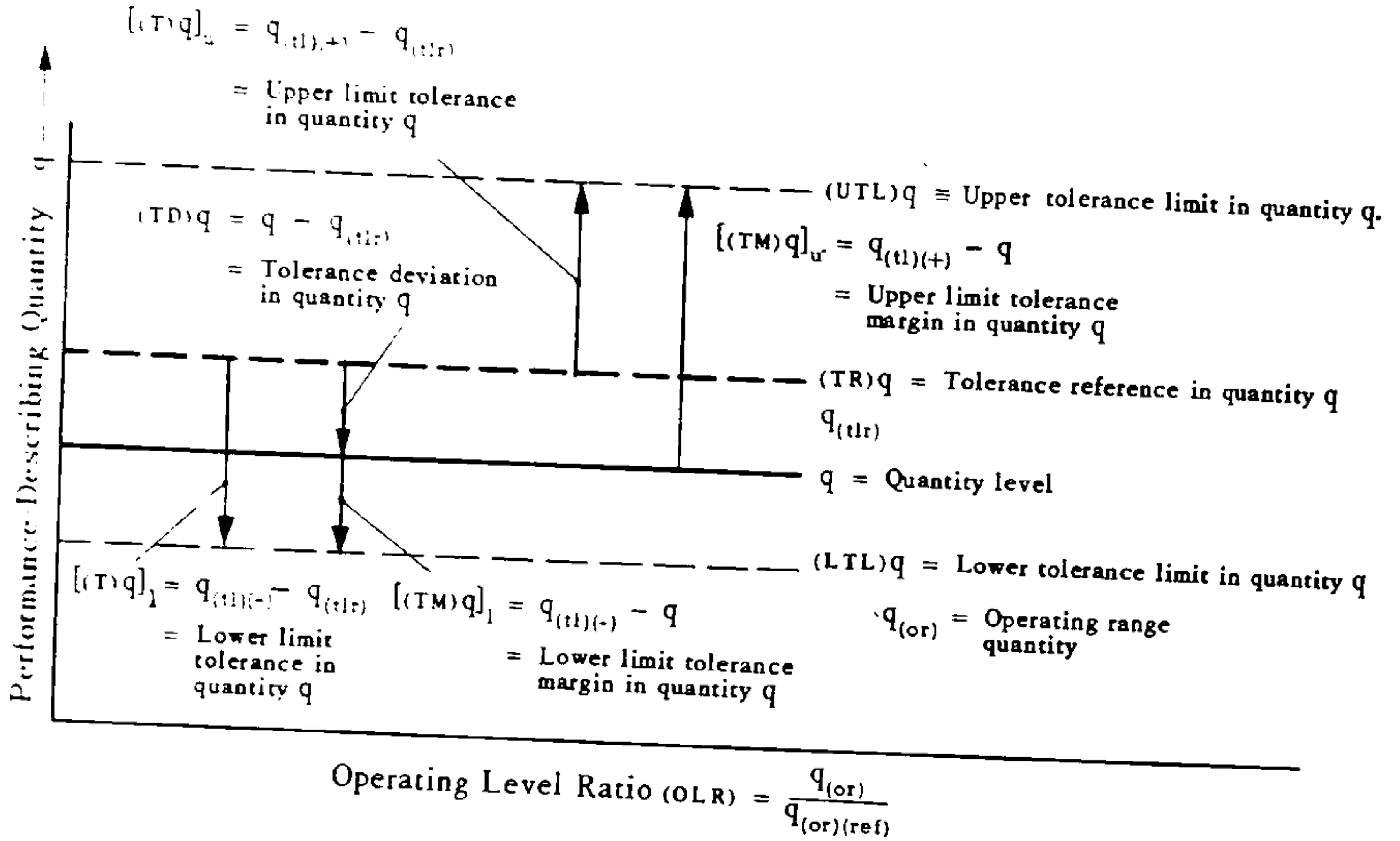
b) Cylinder pressure – time record



c) Cylinder pressure correction and cylinder pressure error curves

Fig. 9-12. Engine cylinder pressure example illustrating corrections and errors.

operation is satisfactory, is taken as the tolerance reference,* (TR)q. The lower tolerance limit, (LTL)q, is the lowest level below the tolerance reference that defines the region of satisfactory operation, while the upper tolerance limit, (UTL)q, defines the region of satisfactory operation above the tolerance reference. The lower limit tolerance in q, $[(T)q]_l$, is the difference between the lower tolerance limit and the tolerance reference. The upper limit tolerance in q, $[(T)q]_u$ is the difference between the upper tolerance limit and the tolerance reference. The difference between any given level of q and the tolerance reference



$[(T)q]_u = q_{(tl)(+)} - q_{(tlr)}$
= Upper limit tolerance in quantity q
$(TD)q = q - q_{(tlr)}$
= Tolerance deviation in quantity q
(UTL)q ≡ Upper tolerance limit in quantity q.
$[(TM)q]_u = q_{(tl)(+)} - q$
= Upper limit tolerance margin in quantity q
(TR)q = Tolerance reference in quantity q
$q_{(tlr)}$
q = Quantity level
(LTL)q = Lower tolerance limit in quantity q
$[(T)q]_l = q_{(tl)(-)} - q_{(tlr)}$
= Lower limit tolerance in quantity q
$[(TM)q]_l = q_{(tl)(-)} - q$
= Lower limit tolerance margin in quantity q
$q_{(or)}$ = Operating range quantity
Performance Describing Quantity — q
Operating Level Ratio (OLR) = $\frac{q_{(or)}}{q_{(or)(ref)}}$

$$[(T)q]_l = (TD)q + [(TM)q]_l \tag{1a}$$
$$= q - q_{(tlr)} + q_{(tl)(-)} - q = q_{(tl)(-)} - q_{(tlr)} \tag{2a}$$
$$= q - (TR)q + (LTL)q - q = (LTL)q - (TR)q \tag{3a}$$
$$[(T)q]_u = (TD)q + [(TM)q]_u \tag{1b}$$
$$= q - q_{(tlr)} + q_{(tl)(+)} - q = q_{(tl)(+)} - q_{(tlr)} \tag{2b}$$
$$= q - (TR)q + (UTL)q - q = (UTL)q - (TR)q \tag{3b}$$

Fig. 9-13. Tolerance-defining quantities.

is the tolerance deviation, (TD)q. The difference between the lower tolerance limit and any given level of q is the lower limit tolerance margin, $[(TM)q]_l$. The corresponding difference for the upper tolerance limit is the upper limit tolerance margin, $[(TM)q]_u$. It is useful to note that for either the upper or lower tolerance limit, the sum of the tolerance deviation and the tolerance margin is always equal to the tolerance itself. All of the tolerance relations defined are shown in Fig. 9-13.

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* The tolerance reference is shown as being constant in Fig. 9-13. This is a special case; in general, the tolerance reference may vary arbitrarily with the operating level ratio.

9.29. Tolerance and its associated concepts are important in all engineering problems. This importance is forced on the engineer because perfection is never possible in practice — the realized performance from any component or system always deviates from the ideal. One of the primary problems of design is to establish the tolerance references required by satisfactory system operation and at the same time to set up tolerance limits that may be held at acceptable cost by available manufacturing procedures. Tolerance considerations appear as essential parts of the examples discussed in Volume III.

CHAPTER 10

CONCEPTS AND NOTATION FOR THE DESCRIPTION OF GENERALIZED OPERATING SYSTEM PERFORMANCE

GENERALIZED PERFORMANCE-DESCRIBING CONCEPTS AND SYMBOLS

10.1. Concepts and notation for describing the performance of operating components and systems that receive a single input and produce a single output are discussed in earlier chapters. Systems of this type represent the simplest possible example of the situations that must be solved by the instrument engineer. In the general case, operating systems are made up of components that cause a number of outputs to vary as functions of a number of inputs. This means that, to be complete, performance-describing concepts and symbols must include provisions for identification of the operating entity with which they are associated and for specification of the inputs and outputs that are considered. The inherent difficulties in manipulating mathematical expressions that contain several independent variables and several dependent variables necessitate forms for describing performance that associate a single output with all of its controlling inputs. In addition to these basic requirements, a satisfactory performance-describing system must also include provisions for representing dimensional relationships among the physical quantities involved; distinctions among inputs of various types; and the relationship of the operating entity whose behavior is represented to the operating subsystems and systems of which it may be a part.

ACTUATING, INTERFERING, AND MODIFYING INPUTS

10.2. Many instrument systems receive inputs that are vector quantities and generate outputs that are also vector quantities. The vector performance operator described in the discussion of the gyroscopic element in Chapter 7 has the basic properties required to describe the behavior of a completely general operating component or system. Figure 10-1 is a functional diagram for an operating component of this type. In this figure, the operating component is identified as belonging to a system with an indefinite number of vector inputs and vector outputs. For practical reasons, the inputs are divided into actuating inputs, interfering inputs, and modifying inputs:

(1) Actuating inputs include those inputs that are of prime importance in the sense that they are essential for operation of the component.

(2) Interfering inputs are received by the operating component and produce output changes that are not directly distinguishable from those due to the actuating inputs. In general, effects due to interfering inputs are undesirable and interfere with the proper response of the operating component to the actuating and modifying inputs.

(3) Modifying inputs change the relationships established by the operating component among the other inputs and the outputs.

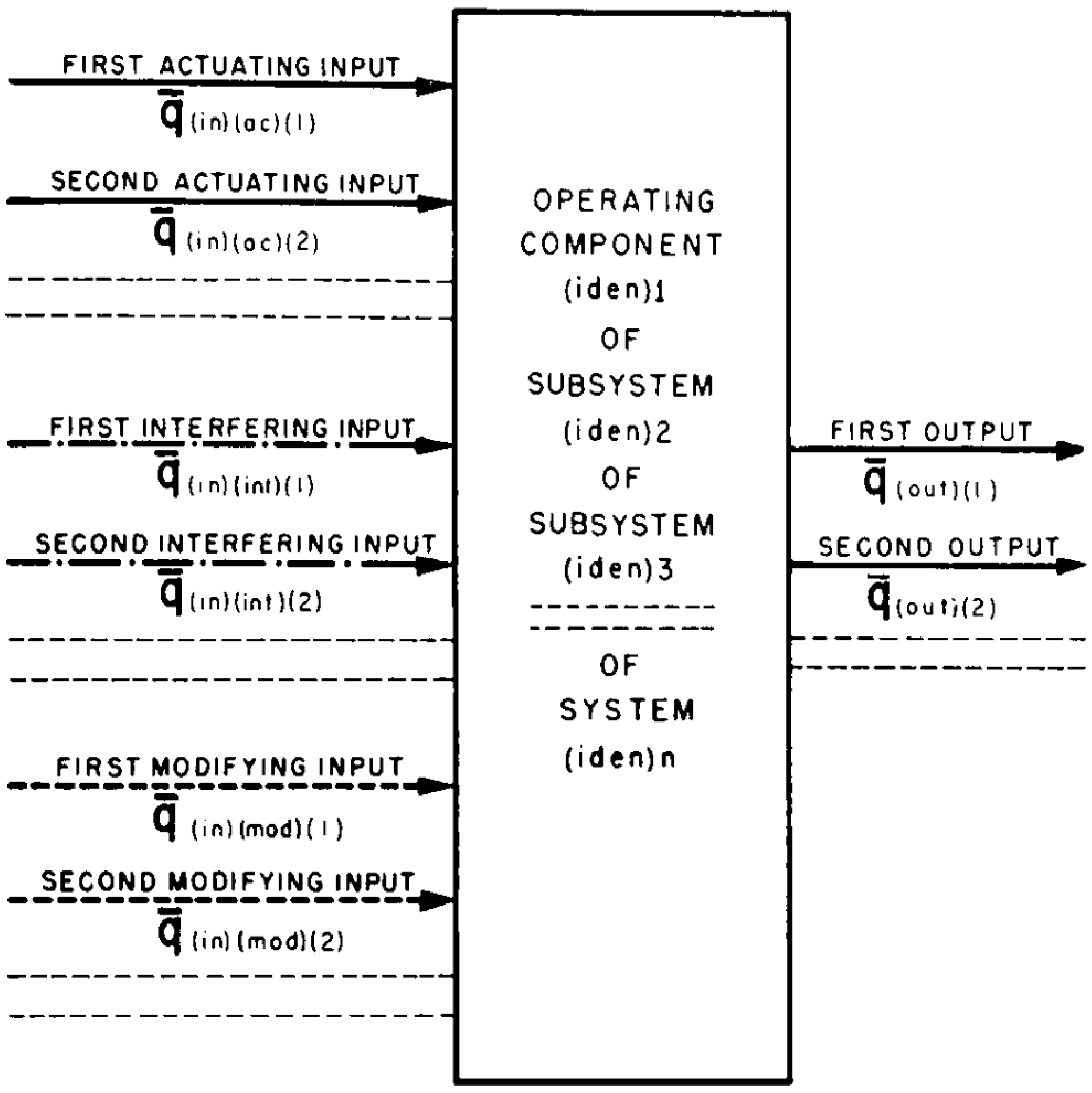
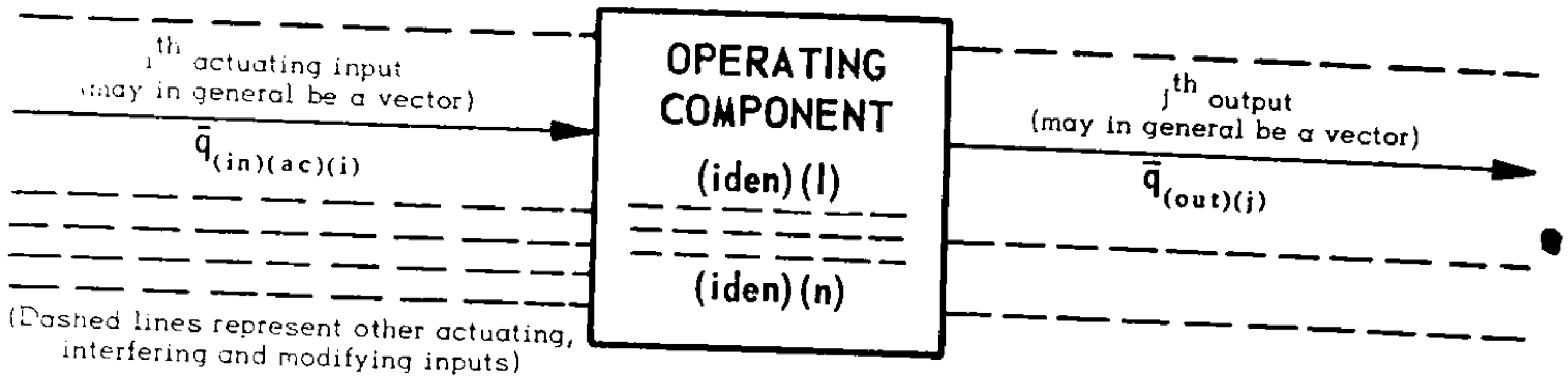


Fig. 10-1. Generalized functional diagram for an operating component with vector inputs and outputs.

SYMBOLS FOR PERFORMANCE-DESCRIBING CONCEPTS

10.3. Figure 10-2 illustrates the general features of a notation for describing the performance of operating components and the inputs and outputs associated with their operation. A series of identifying symbols, written as (iden)(1), (iden)(2), . . . etc., is associated with the operating component itself and the subsystems to which it belongs. For purposes of formal distinction, the first identification symbol is associated directly with the operating component; the next symbol in order represents the simplest subsystem of which the operating component is a part; the third identification symbol denotes the next larger subsystem of which the simplest subsystem is a part. This general scheme is continued until the final identification symbol represents the over-all system that contains


i th actuating input (may in general be a vector)
$\bar{q}_{(in)(ac)(i)}$
(Dashed lines represent other actuating, interfering and modifying inputs)
OPERATING COMPONENT
(iden)(1)
(iden)(n)
j th output (may in general be a vector)
$\bar{q}_{(out)(j)}$

a) Simplified functional diagram for a selected input-output pair associated with an operating component requiring n identifications for complete specification

The generalized mathematical expression for the relationship between an input-output pair that may in general be vectors is as follows:

$$\bar{q}_{(out)(j)} = (\bar{f}_{(j)(mod)})_{(iden)(1),\ldots[q_{(in)(ac)(i)};q_{(out)(j)}][(iden)(n)]}(\bar{q}_{(in)(ac)(i)})$$

where the bars over the quantities and the function symbol indicate vectors or vector functions.

Identification subscript symbols start at the left (nearest to the main symbol) with a direct representation of the operating component; the next identification symbol designates the subsystem of which the operating component is a part, and so on until identification is complete. This general pattern of subscript location may be varied somewhat for special situations, but will simplify the recognition of symbols if it is used consistently.

The subscript representing the input quantity is placed within a square bracket, in the "reference quantity" position, which is nearest the main symbol. The use of this position shows that the input is an independent quantity so far as being directly affected by the operating component is concerned.

The subscript representing the output quantity is placed within the input-output bracket in the "compared quantity" position, to the right of the input quantity and separated from it by a semi-colon.

The n^{th} identifying subscript is shown after the input-output bracket to illustrate the possibility of using this location. Ordinarily, it would be used to differentiate between substantially like subsystems used as parts of the same over-all system.

b) Generalized mathematical expression for the relationship between an input-output pair that may in general be vectors

Fig. 10-2. Representation of operating system functional performance in functional relationship form.

all the identified subsystems. In practice, the identification symbols are either grouped together to represent a given operating component directly, or they are used to form a system of subscript symbols to identify performance-describing concepts associated with the identified operating component.

REPRESENTATION OF INPUT AND OUTPUT QUANTITIES

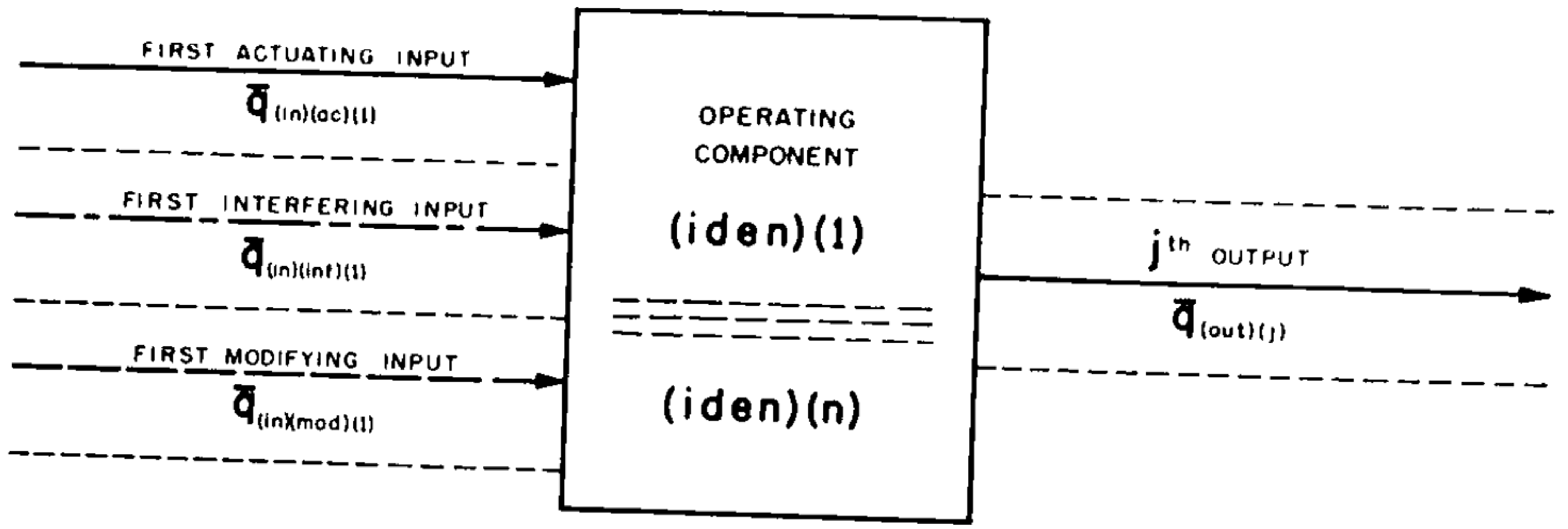
10.4. In Fig. 10-2, the small letter q is used to represent a generalized physical quantity. A subscript (in) added to this symbol indicates that the physical quantity considered belongs to the class of inputs. An additional subscript, either (ac), (int), or (mod), following (in), shows that the identified quantity belongs to the class of actuating inputs, interfering inputs, or modifying inputs, respectively. Where more than one input of these subclasses exists, the necessary distinction between individual inputs is made by numbers placed after the classifying subscripts.

10.5. A subscript (out) on the physical quantity symbol shows that the identified quantity belongs to the general class of outputs. Numbers following the (out) subscript distinguish between outputs when more than one is associated with a single operating component.

10.6. It is possible to identify physical quantities not only as inputs or outputs but also as being associated with a given operating component. This identification is made by adding the proper group of subscripts to represent the operating component to the subscripts for the physical quantities involved. Except for special cases, the circumstances surrounding the use of symbols for input and output quantities define without ambiguity the operating component with which they are associated. For this reason, operating component identification is ordinarily omitted from input and output quantity symbols. Extensions of the principles described here may be applied at any time to completely identify physical quantity concepts whenever this identification is required. Many examples of this principle appear later in this book.

NOTATIONAL CONVENTIONS FOR THE PERFORMANCE OPERATOR

10.7. Figure 10-3 summarizes the notational conventions for the performance operator, which is partially defined in earlier chapters. This concept is one of the basic tools of instrument engineering. The capital letters PO, enclosed by two square brackets, represent the performance operator. The bar over the P shows that the operator considered has vector properties. The series of subscript symbols (iden)(1) . . ., starting adjacent to the right-hand operator bracket, identifies the operating component whose performance, in functionally relating the inputs to a specified output, is described by the operator. The series of symbols in the subscript just inside the left-hand square bracket that are separated by commas specifies the physical quantities that act as actuating inputs. Following a semicolon after the actuating inputs, the interfering inputs are listed with separating commas. A second semicolon indicates the start of the series of symbols that specifies the modifying inputs. The symbol following a third semicolon represents the output quantity. For illustrative purposes, in Fig. 10-3 the output is taken as the j^{th} output quantity. The square bracket of the subscript is closed after the output symbol, and a final identification symbol (that may or may not be required in a given case) completes the subscript. As it is described above, the performance operator is intended to represent completely the functional action of an identified operating component in establishing functional relationships between its inputs and a selected output. One performance operator of this general form is required for each of the outputs associated with a given operating component.



FOR GENERALITY ALL INPUTS AND OUTPUTS ARE SHOWN AS VECTORS
IN PRACTICE SOME OR ALL OF THE QUANTITIES MAY BE SCALARS

a) Functional diagram for generalized operating component

$$\bar{q}_{(out)(j)} = [\overline{PO}]_{(iden)(1)\cdots[q_{(in)(ac)(1)},\cdots;q_{(in)(int)(1)},\cdots q_{(in)(mod)(1)},\cdots;q_{(out)(j)}](iden)(n)}[\bar{q}_{(in)(ac)(1)},\cdots;\bar{q}_{(in)(int)(1)},\ \cdots;\bar{q}_{(in)(mod)(1)}\ \ldots]$$

b) Generalized vector performance equation for j^{th} output of completely identified operating component

$[\overline{PO}]$	(iden)(1)···	$[q_{(in)(ac)(1)},\cdots$;	$q_{(in)(int)(1)},\cdots$;	$q_{(in)(mod)(1)},\cdots$;	$q_{(out)(j)}]$	(iden)(n)
MAIN SYMBOL SHOWING THAT THE CONCEPT REPRESENTED IS A PERFORMANCE OPERATOR BAR OVER SYMBOL SHOWS THAT A VECTOR RELATIONSHIP IS INVOLVED [ ] SQUARE BRACKETS SHOW THAT SYMBOL DESCRIBES PERFORMANCE OF AN OPERATING COMPONENT.	IDENTIFYING SUBSCRIPTS STARTING WITH COMPONENT ITSELF AND PROGRESSING TOWARD SUB-SYSTEMS OF GREATER COMPLEXITY, INCLUDING THE SUB-SYSTEM IDENTIFIED BY SYMBOLS TO LEFT IDENTIFICATION SUBSCRIPT SYMBOLS ARE ENCLOSED BY PARENTHESES.	ACTUATING INPUTS, SEPARATED BY COMMAS	INTERFERING INPUTS, SEPARATED BY COMMAS	MODIFYING INPUTS SEPARATED BY COMMAS	DESIGNATED OUTPUT	FINAL IDENTIFICATION OF MOST INCLUSIVE SYSTEM DESIGNATED (USED ONLY WHERE NECESSARY)

THE INPUT-OUTPUT SUBSCRIPT IS ENCLOSED BY SQUARE BRACKETS. THE THREE CLASSES OF INPUTS ARE SEPARATED BY SEMICOLONS. THE INPUT GROUP IS SEPARATED FROM THE OUTPUT BY A SEMICOLON. NOTE THAT THE FIRST ACTUATING INPUT IS ALWAYS ADJACENT TO THE INSIDE OF THE LEFT-HAND SQUARE BRACKET. THE DESIGNATED OUTPUT SYMBOL IS ALWAYS ADJACENT TO THE RIGHT-HAND SQUARE BRACKET.

c) Generalized vector performance operator for j^{th} output of completely identified operating component

NOTE:. If vector quantities are not included among the input and output quantities, the vector performance operator reduces to the scalar performance operator. This operator symbol is identical with the vector performance operator except that the superscript bar is omitted.

Fig. 10-3. Conventions for forming vector performance operator symbols with complete subscript system.

10.8. Vector performance operators formed under the conventions of Fig. 10-3 are useful and convenient for generalized descriptions of operating system behavior and in some special cases may be adapted for carrying out all the theoretical details for a design. However, many operating components carry out such operations as multiplication, division, and algebraic addition only by scalar processes, so that vector performance operators must often be resolved into component scalar performance operators. Section a) of Definition Summary 10-1 shows that the scalar performance operator is identical in form with the vector performance operator except that the bar over the PO inside the operator bracket is omitted. The conventions for the formation of subscripts that have already been explained for vector performance operators are applied to scalar performance operators.

$[\mathrm{PO}]_{(\mathrm{iden})(1),\ldots[q_{(\mathrm{in})(\mathrm{ac})(1)},\ldots;\ q_{(\mathrm{in})(\mathrm{int})(1)},\ldots;\ q_{(\mathrm{in})(\mathrm{mod})(1)},\ldots;\ q_{(\mathrm{out})(j)}](\mathrm{iden})(n)} \equiv$ j^{th} Output Scalar Performance Operator for (iden)(1) ... Identified Operating System

The scalar performance operator represents the scalar action on the j^{th} output of the operation of the identified operating component when the effects due to all the essential inputs are considered.

Subscript conventions for scalar performance operators are identical with the conventions for vector performance operators as specified in Fig. 10-3.

a) Generalized scalar performance operator for j^{th} output of identified operating component

$[\mathrm{PF}]_{(\mathrm{iden})(1),\ldots[q_{(\mathrm{in})(\mathrm{ac})(1)},\ldots;\ q_{(\mathrm{in})(\mathrm{int})(1)},\ldots;\ q_{(\mathrm{in})(\mathrm{mod})(1)},\ldots;\ q_{(\mathrm{out})(j)}](\mathrm{iden})(n)} \equiv$ j^{th} Output Performance Function for (iden)(1)... Identified Operating System

The performance function is a special form of the scalar performance operator. It is a scalar function with the special property that a series of operators representing the performance of an open-chain combination of operating components may be used in the expression for over-all performance without regard to order.

Performance functions often have product-type combinations of input quantities as their arguments.

In general, more than one performance function with its corresponding input function product argument will be required to completely describe the j^{th} output performance of an operating component.

Subscript conventions for performance functions are identical with the conventions for vector performance operators as specified in Fig. 10-3.

The form used for the enclosing symbols of the input-output subscript is especially important.

1) When square brackets are used, i.e., [ ], the performance operator is complete, carrying with it all pertinent dimensions and an algebraic sign.
2) When straight vertical lines are used, i.e., | |, the performance operator carries all pertinent dimensions, but only the magnitude of the operator is represented.
3) When conventional curved brackets, are used, i.e., ( ), the subscript is for identification purposes only – no dimensions are associated with the operator.

b) Generalized performance function definition and notation conventions

Definition Summary 10-1. Scalar performance operator and performance function.

$$q_{(out)(j)}(q_{(in)(ac)(i)} \ldots q_{(in)(ac)(k)}) = [PF]_{(iden)(1),\ldots(q_{(in)(ac)(1)},\ldots q_{(in)(ac)(k)};q_{(in)(mod)(1)}\ldots;q_{(out)(j)}((iden)(n)} \times [f_{(ac)(i)}(q_{(in)(ac)(i)})\ldots f_{(ac)(k)}(q_{(in)(ac)(k)})f_{(mod)(1)}(q_{(in)(mod)(1)})\ldots]$$

= ith ... kth actuating inputs component of jth output of (iden)(1) ... identified operating component

where

$f_{(ac)(i)}(q_{(in)(ac)(i)})$ ≡ function of ith actuating input;

.

.

.

$f_{(ac)(k)}(q_{(in)(ac)(k)})$ ≡ function of kth actuating input;

$f_{(mod)(1)}(q_{(in)(mod)(1)})$ ≡ function of first modifying input

.

.

.

It is understood without addition of subscript symbols on the output component that all effects due to the essential modifying inputs are properly taken into account by the forms of the actuating input functions.

Corresponding relationships for components of the jth output exist for a sufficient number of actuating input function products to completely account for all the input-output functional relationships that exist because of the action of the (iden)(1) ... identified operating component.

Components of the jth output exist for interfering input function products similar to the components corresponding to the actuating input function product components. The symbols representing these interfering input function product components are identical with the corresponding symbols for the actuating input function product components except that the subscript symbols (ac) are replaced by the subscript symbols (int).

a) Representation of one component of the jth output of an identified operating system by means of a single performance functio.. with a single input function product argument.

$$q_{(out)(j)} = [q_{(out)(j)}]_{[q_{(in)(ac)(1)}\ldots q_{(in)(ac)(k)}]} + \ldots + [q_{(out)(j)}]_{[q_{(in)(int)(1)}\ldots q_{(in)(int)(p)}]} + \ldots$$

The jth output of the (iden)(1) ... identified operating component can always be represented by a sum of output components, each of which depends upon a performance function and the corresponding input function product. This series of terms must be extended until the combined effects of all inputs are completely taken into account.

b) Complete representation of the jth output from an (iden)(1) ... identified operating component as a sum of input function product components.

Definition Summary 10-2. Representation of the jth output from an (iden)(1) ... identified operating component in terms of performance functions and input function products.

PERFORMANCE FUNCTIONS

10.9. A special form of the scalar performance operator is the performance function; this concept is represented by the main symbol [PF], with subscripts added according to the conventions described in Fig. 10-3. By definition, performance functions are scalar functions that may be combined without regard to order for the representation of performance for chains of operating components. Section b) of Definition Summary 10-1 gives the generalized symbol for the performance function and illustrates the use of a set of subscripts for identification purposes.

10.10. Many operating components establish relationships between product-type combinations of functions that depend upon several inputs. This fact makes

The effect of all actuating inputs on the j^{th} output may be written

$$q_{(out)(j)} = [PF]_{(iden)(1),\ldots[q_{(in)(ac)(i)},\ldots q_{(in)(ac)(k)};\, q_{(out)(j)}](iden)(n)}$$

$$\times\ [f_{(ac)(i)}(q_{(in)(ac)(i)})\ldots f_{(ac)(k)}(q_{(in)(ac)(k)}) f_{(mod)(1)}(q_{(in)(mod)(1)})\ldots] \ +$$

$$\ldots + [PF]_{(iden)(1),\ldots[q_{(in)(int)(o)},\ldots q_{(in)(int)(p)};\, q_{(out)(j)}](iden)(n)}$$

$$\times\ [f_{(int)(o)}(q_{(in)(int)(o)})\ldots f_{(int)(p)}(q_{(in)(int)(p)})\ldots f_{(mod)(1)}(q_{(in)(mod)(1)})\ldots] \ + \ldots$$

or, in summation form

$$q_{(out)(j)} = \sum [PF]_{(iden)(1),\ldots[q_{(in)(ac)(o)},\ldots q_{(in)(ac)(k)};\, q_{(out)(j)}](iden)(n)}$$

Sum taken over the number of terms required to introduce effects of all actuating and interfering inputs on the j^{th} output

$$\times\ [f_{(ac)(o)}(q_{(in)(ac)(1)})\ldots f_{(ac)(k)}(q_{(in)(ac)(k)})\ldots f_{(mod)(1)}(q_{(in)(mod)(1)})\ldots]$$

a) Representation of the actuating and modifying inputs on the j^{th} output of an (iden)(1)... identified operating system in terms of formally extended input function products.

$$q_{(out)(j)} = \sum [PF]_{(iden)(1),\ldots[q_{(in)(ac)(i)},\ldots q_{(in)(ac)(k)};\, q_{(out)(j)}](iden)(n)} (IFP)_{(ac)(i\ldots k)}$$

Sum taken over the number of terms required to introduce effects of all actuating inputs on the j^{th} output.

$$+ \sum [PF]_{(iden)(1),\ldots[q_{(in)(int)(o)},\ldots q_{(in)(int)(p)};\, q_{(out)(j)}](iden)(n)} (IFP)_{(int)(o\ldots p)}$$

Sum taken over the number of terms required to introduce effects of all interfering inputs on the j^{th} output.

where by definition

$$(IFP)_{(ac)(i\ldots k)} = [f_{(ac)(i)}(q_{(in)(ac)(i)})\ldots f_{(ac)(k)}(q_{(in)(ac)(k)}) f_{(mod)(1)}(q_{(in)(mod)(1)})\ldots]$$

$\equiv$ i^{th} to k^{th} actuating input function product

$$(IFP)_{(int)(o\ldots p)} = [f_{(int)(o)}(q_{(in)(int)(o)})\ldots f_{(int)(p)}\,(q_{(in)(int)(p)}) f_{(mod)(1)}(q_{(in)(mod)(1)})\ldots]$$

$\equiv$ o^{th} to p^{th} interfering input function product

Input function products may contain all three types of inputs when this is necessary to describe the effect of any input on the j^{th} output of the (iden)(1)... identified operating component.

b) Representation of the actuating, modifying and interfering inputs on the j^{th} output of an (iden)(1)... identified operating system with the input function products replaced by the symbol (IPF), with appropriate subscripts.

Definition Summary 10-3. Formal representation of the j^{th} output from an (iden)(1)... identified operating component in terms of performance functions and input function products.

it convenient to use combinations of inputs as entities in the practical use of performance operators. In general, a selected output from an operating component may not be expressible solely in terms of a single performance function and one input function product argument. This means that the complete expression for the j^{th} output of an operating component may naturally break down into parts, or components, each of which is controlled by a different input function product and a different performance function form. Section a) of Definition Summary 10-2 gives an illustrative expression, complete with identifying subscripts, for the i^{th} to k^{th} actuating inputs component of the j^{th} output. Section b) of this definition summary shows the j^{th} output expressed as a sum of input function product components with the inputs involved written as subscripts to identify the individual output components.

10.11. Section a) of Definition Summary 10-3 shows the expanded form of the expression for the j^{th} output from an identified operating component written as two summations, one for components controlled primarily by actuating inputs and one for components controlled primarily by interfering inputs. In order to simplify the form, the explicit expressions for the input function products of section a) of the definition summary are replaced by the symbol (IFP) (initials of the words input function product) with the subscript (ac) or (int), for either actuating or interfering input, and a set of symbols identifying the specific inputs involved.

PERFORMANCE FUNCTION RATIO FUNCTIONS

10.12. Instrument engineering problems may involve operating components that have several inputs of comparable importance. It is ordinarily convenient to select one of the inputs as the reference input and to express the effects of all the other inputs on the output under consideration in terms of this reference input. This need for expressing the effect of one input in terms of another input leads to the concept of the performance function ratio function,* which is given the main symbol (PFRF) and carries a system of subscripts based on the conventions summarized in Fig. 10-3.

10.13. In Definition Summary 10-4, section a) shows in symbols and in words the relationships between the performance function ratio function and the concepts defined in previous figures. Reduced to its simplest terms, the procedure for forming performance function ratio functions is to select one input function product as the reference input function product and to divide each term of the set required to completely express the j^{th} output by the reference performance function, which is the function in which the reference input function product appears as the argument. The reference performance function is then taken outside of a bracket which contains a term consisting of the reference input function product and two summations. One of these summations is for the actuating inputs and one is for the interfering inputs, their terms being made up of performance function ratio functions with the corresponding input function products. The resulting form is given in Eq. (1) of Definition Summary 10-4.

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* The term ratio function is used to describe the result of "dividing" one performance function by a second performance function. If the entities involved were scalars, the "division" would produce a ratio with the nature of another scalar. The use of the term ratio function instead of ratio in the terminology involved here recognizes the fact that the specified combination of performance functions will, in general, result in a function rather than a simple scalar ratio.

$$q_{(out)(j)} = [\mathrm{PF}]_{(iden)(1),\ldots[q_{(in)(ac)(ref)(1)},\ldots q_{(in)(ac)(ref)(r)};q_{(out)(j)}](iden)(n)}$$

$$\times \Big\{(\mathrm{IFP})_{(ac)(ref)(1\text{-}r)} + \Sigma\,(\mathrm{PFRF})_{(iden)(1),\ldots[(IFP)_{(ac)(i\text{-}k)};(IFP)_{(ac)(ref)(1\text{-}r)}](iden)(n)}(\mathrm{IFP})_{(ac)(i\text{-}k)}$$

Sum over number of terms required to introduce effects of all actuating inputs on the j^{th} output except those in (1-r) actuating input function reference product.

$$+ \Sigma\,(\mathrm{PFRF})_{(iden)(1),\ldots[(IFP)_{(int)(\ell\text{-}m)};(IFP)_{(ac)(ref)(1\text{-}r)}](iden)(n)}(\mathrm{IFP})_{(int)(\ell\text{-}m)}\Big\}$$

Sum over number of terms required to introduce effects of all interfering inputs on the j^{th} output.

(1)

where by definition

$$(\mathrm{PFRF})_{(iden)(1),\ldots[(IFP)_{(ac)(i\text{-}k)};(IFP)_{(ac)(ref)(1\text{-}r)}](iden)(n)} = \frac{[\mathrm{PF}]_{(iden)(1),\ldots[q_{(in)(ac)(i)},\ldots q_{(in)(ac)(k)};q_{(out)(j)}](iden)(n)}}{[\mathrm{PF}]_{(iden)(1),\ldots[q_{(in)(ac)(ref)(1)},\ldots q_{(in)(ac)(ref)(r)};q_{(out)(j)}](iden)(n)}}$$

$\equiv$ i^{th} and k^{th} actuating input function product – reference input function product performance function ratio functions. (2)

$$(\mathrm{PFRF})_{(iden)(1),\ldots[(IFP)_{(int)(\ell\text{-}m)};(IFP)_{(ac)(ref)(1\text{-}r)}](iden)(n)} = \frac{[\mathrm{PF}]_{(iden)(1)\ldots[q_{(in)(int)(\ell)},\ldots q_{(in)(int)(m)};q_{(out)(j)}](iden)(n)}}{[\mathrm{PF}]_{(iden)(1)\ldots[q_{(in)(ac)(ref)(1)},\ldots q_{(in)(ac)(ref)(r)};q_{(out)(j)}](iden)(n)}}$$

$\equiv$ ℓ^{th} to m^{th} interfering input function product – reference input function product performance function ratio functions. (3)

a) Definitions and symbols for performance function ratio functions

Performance function ratio functions show the relative effects of various input function products on a specified output as compared with the effect of the input function product chosen as the reference input function product. In formal symbols, this statement depends upon the relationships:

$$\frac{[q_{(out)(j)}]_{(IFP)_{(ac)(i\text{-}k)}}}{[q_{(out)(j)}]_{(IFP)_{(ac)(ref)(1\text{-}r)}}} = \frac{[\mathrm{PF}]_{(iden)(1),\ldots[q_{(in)(ac)(i)},\ldots q_{(in)(ac)(k)};q_{(out)(j)}](iden)(n)}(\mathrm{IFP})_{(ac)(i\text{-}k)}}{[\mathrm{PF}]_{(iden)(1),\ldots[q_{(in)(ac)(ref)(1)},\ldots q_{(in)(ac)(ref)(r)};q_{(out)(j)}](iden)(n)}(\mathrm{IFP})_{(ac)(ref)(1\text{-}r)}}$$

$$= (\mathrm{PFRF})_{(iden)(1),\ldots[(IFP)_{(ac)(i\text{-}k)};(IFP)_{(ac)(ref)(1\text{-}r)}](iden)(n)}\left\{\frac{(\mathrm{IFP})_{(ac)(i\text{-}k)}}{(\mathrm{IFP})_{(ac)(ref)(1\text{-}r)}}\right\} \quad (4)$$

so that

$$(\mathrm{PFRF})_{(iden)(1),\ldots[(IFP)_{(ac)(i\text{-}k)};(IFP)_{(ac)(ref)(1\text{-}r)}](iden)(n)} = \left\{\frac{[q_{(out)(j)}]_{(IFP)_{(ac)(i\text{-}k)}}}{[q_{(out)(j)}]_{(IFP)_{(ac)(ref)(1\text{-}r)}}}\right\}\left\{\frac{(\mathrm{IFP})_{(ac)(ref)(1\text{-}r)}}{(\mathrm{IFP})_{(ac)(i\text{-}k)}}\right\} \quad (5)$$

For the special case of equal effects from the $(i\text{-}k)^{th}$ input function product and the reference input function product; i.e., when

$$[q_{(out)(j)}]_{(IFP)_{(ac)(i\text{-}k)}} = [q_{(out)(j)}]_{(IFP)_{(ac)(ref)(1\text{-}r)}}$$

then

$$\Big[(\mathrm{IFP})_{(ac)(ref)(1\text{-}r)}\Big]\Big\{[q_{(out)(j)}]_{(IFP)_{(ac)(i\text{-}k)}} = [q_{(out)(j)}]_{(IFP)_{(ac)(ref)(1\text{-}r)}}\Big\}$$

$$= (\mathrm{PFRF})_{(iden)(1)\ldots[(IFP)_{(ac)(i\text{-}k)};(IFP)_{(ac)(ref)(1\text{-}r)}](iden)(n)}(\mathrm{IFP})_{(ac)(i\text{-}k)}$$

or

(6)

$$(\mathrm{PFRF})_{(iden)(1),\ldots[(IFP)_{(ac)(i\text{-}k)};(IFP)_{(ac)(ref)(1\text{-}r)}](iden)(n)} = \left\{\frac{(\mathrm{IFP})_{(ac)(ref)(1\text{-}r)}}{(\mathrm{IFP})_{(ac)(i\text{-}k)}}\right\}\Big\{[q_{(out)(j)}]_{(IFP)_{(ac)(i\text{-}k)}} = [q_{(out)(j)}]_{(IFP)_{(ac)(ref)(1\text{-}r)}}\Big\}$$

In words:

(7)

$$\left.\begin{array}{l}\text{The } i^{th} \text{ to } k^{th} \text{ actuating input function product – reference}\\ \text{input function product performance function ratio function}\end{array}\right\} = \left\{\begin{array}{l}\text{Ratio of reference input function product to } i^{th} \text{ to } k^{th}\\ \text{function product for equal effects on the } j^{th} \text{ output}\end{array}\right.$$

If interfering inputs are considered, instead of actuating inputs, the equations above remain unchanged in general form, but subscripts (ac) must be replaced by subscripts (int).

b) Relationship of performance function ratio function to the effect on the output from a specified input function product and to the corresponding output effect due to the reference input function product

Definition Summary 10-4. Definition of performance function ratio function and relationship of this quantity to the inputs of an operating component required to produce equal output effects.

10.14. Section b) of Definition Summary 10-4 develops the relationships between the performance function ratio functions and the relative effects of input function products on the output. The basic expression given in Eq. (4) of this definition summary is the ratio of the i^{th} to k^{th} actuating input function product component of the j^{th} output component to the reference input function product component of this output component. When the ratio of the performance functions is written as a properly identified performance function ratio function, it appears in Eq. (5) of Definition Summary 10-4 that this function is equal to the ratio of the i^{th} to k^{th} actuating input output component to the reference input output component, multiplied by the ratio of the reference input function product to the i^{th} to k^{th} actuating input function product. For the special case in which the output components corresponding to the two input function products are equal, it appears that, from the standpoint of physical fact, the i^{th} to k^{th} actuating input function product – reference input function product performance function ratio function reduces to the ratio of the reference input function product to the i^{th} to k^{th} input function product. This fact, which is expressed in words and by Eq. (7) of Definition Summary 10-4, provides a simple method for measuring performance function ratio functions for actual equipment. It is only necessary in practice to apply a known value of the reference input function product for the operating system considered and then to counterbalance its effect on the selected output by means of the other input function product involved. The ratio between the two input function products is then a direct measure of the corresponding performance function ratio function.

SENSITIVITY AND TIME RESPONSE FUNCTION

10.15. In practice, it is generally useful to separate performance functions into components of two types. One of these components carries all the necessary dimensions (with the possible exception of time), while the other represents time variation effects introduced by the operating component between the inputs and a selected output. The first component of the performance function is the sensitivity, which is defined in paragraph 4.4 of Chapter 4. The second component of the performance function is called the time response function, and is given (TRF) as its main symbol. Both sensitivities and time response functions carry a system of subscripts formed by the conventions described in Fig. 10-3. In any given example, the sensitivity is ordinarily chosen to account so far as possible for magnitude effects between inputs and outputs. In the same way, time response functions are chosen to describe response delay, which is defined in paragraph 6.3 of Chapter 6 as the difference between the instant a given value of an input function product occurs and the instant at which the corresponding value of the output appears. Section a) of Definition Summary 10-5 brings together in words and symbols [Eq. (1)] the relationships between the performance function and its sensitivity and time-response-function components.

$$[PF]_{(iden)(1),\ldots[q_{(in)(ac)(i)},\ldots,q_{(in)(ac)(k)};q_{(in)(mod)(1)}\ldots;q_{(out)(j)}](iden)(n)}$$
$$= S_{(iden)(1),\ldots[q_{(in)(ac)(i)},\ldots,q_{(in)(ac)(k)};q_{(in)(mod)(1)}\ldots;q_{(out)(j)}](iden)(n)}$$
$$\times (TRF)_{(iden)(1),\ldots[q_{(in)(ac)(i)},\ldots q_{(in)(ac)(k)};q_{(in)(mod)(1)},\ldots;q_{(out)(j)}](iden)(n)} \tag{1}$$

where by definition

$$S_{(iden)(1),\ldots[q_{(in)(ac)(i)},\ldots,q_{(in)(ac)(k)};q_{(in)(mod)(1)}\ldots;q_{(out)(j)}](iden)(n)}$$

≡ i^{th} to k^{th} actuating input j^{th} output sensitivity of (iden)(1)...identified operating component

$$(TRF)_{(iden)(1),\ldots[q_{(in)(ac)(i)},\ldots,q_{(in)(ac)(k)};q_{(in)(mod)(1)}\ldots;q_{(out)(j)}](iden)(n)}$$

≡ i^{th} to k^{th} actuating input j^{th} output time response function of (iden)(1)...identified operating component

By definition, both sensitivity and time response function, in general, depend upon the way inputs vary with time, and

1. The sensitivity has all dimensions that are associated with the performance ratio except (possibly) time.
2. The time response function may have dimension of time or it may be dimensionless.
3. The sensitivity is a scalar factor relating magnitudes of the output to the corresponding magnitudes of the input.
4. The time response function relates the instant a given input magnitude occurs to the instant the corresponding output magnitude occurs.

a) Definitions of sensitivity and time response function

$$[PF]_{(iden)(1),\ldots[q_{(in)(ac)(i)},\ldots,q_{(in)(ac)(k)};q_{(in)(mod)(1)},\ldots;q_{(out)(j)}](iden)(n)(ref)}$$
$$= S_{(iden)(1),\ldots[q_{(in)(ac)(i)},\ldots q_{(in)(ac)(k)};\, q_{(in)(mod)(1)},\ldots;\, q_{(out)(j)}](iden)(n)(ref)}$$
$$\times (TRF)_{(iden)(1),\ldots[q_{(in)(ac)(i)},\ldots q_{(in)(ac)(k)};q_{(in)(mod)(1)},\ldots;q_{(out)(j)}](iden)(n)(ref)} \tag{2}$$

where

$$S_{(iden)(1),\ldots[q_{(in)(ac)(i)},\ldots q_{(in)(ac)(k)};q_{(in)(mod)(1)},\ldots;q_{(out)(j)}](iden)(n)(ref)}$$

≡ i^{th} to k^{th} actuating input j^{th} output reference sensitivity of (iden)(1),...identified operating component

$$(TRF)_{(iden)(1),\ldots[q_{(in)(ac)(i)},\ldots q_{(in)(ac)(k)};q_{(in)(mod)(1)},\ldots;q_{(out)(j)}](iden)(n)(ref)}$$

≡ i^{th} to k^{th} actuating input j^{th} output time response reference function of (iden)(1),...identified operating component

By definition

1. The reference sensitivity has all the dimensions associated with the performance ratio except (possibly) time.
2. The referense sensitivity is a special constant or varying form of the sensitivity corresponding to some convenient pattern of operating component performance.
3. The time response reference function is an arbitrarily chosen example of the time response function.

b) Definitions of reference sensitivity and time response reference function

Definition Summary 10-5. Definitions of sensitivity and time response function.

10.16. Sensitivities and time response functions are readily adapted to the description of relative effects produced by various input function products. In formal terms, comparisons of this kind are made by the use of a reference sensitivity and its companion time response reference function. By definition, the reference sensitivity is chosen in such a way that it corresponds to some convenient pattern of operating component performance. This characteristic distinguishes the reference sensitivity from the sensitivity as it is defined in section a) of Definition Summary 10-5.

10.17. Because of its special nature, the time response reference function will not be the same as the time response function appearing in the general form of a given performance function. The definitions and symbols associated with the reference sensitivity and the time response reference function are given in section b) of Definition Summary 10-5.

SENSITIVITY RATIO AND TIME RESPONSE FUNCTION RATIO FUNCTION

10.18. The performance function ratio function as defined in Definition Summary 10-4 has its counterpart in a ratio and a ratio function:

(1) The sensitivity ratio, main symbol (SR), which is the ratio of the reference sensitivity corresponding to a given input function product to the reference sensitivity corresponding to the selected reference input function product.

(2) The time response function ratio function, main symbol (TRFRF), which is the ratio of the time response reference function corresponding to a given input function product to the time response reference function corresponding to the selected reference input function product.

10.19. Definition Summary 10-6 gives in words and symbols the essential information on sensitivity ratios, time response function ratio functions, and their relationships to the performance function for an identified operating component.

10.20. By definition, the complete performance function is equivalent to a summation of all the performance functions that correspond to all the input function products that act as their arguments. By the rules of Definition Summary 10-5, each one of the component performance functions may be written as the product of a reference sensitivity and a time response reference function. In order to form sensitivity ratios and time response function ratio functions, it is first necessary to select one of the input function products as the reference input function product. The reference sensitivity and the time response reference function corresponding to this reference input function product are then divided into each term of the set making up the complete performance function (with its system of subscripts built up by the conventions of Fig. 10-3) as given in Definition Summary 10-6. The reference sensitivity and the time response reference function for the reference input function product appear outside of a large bracket. The first term inside this large bracket is an empty bracket with (ac)(ref)(1-r) as its subscript, showing that the reference input function product, which is made up of actuating inputs designated as 1 to r, is to be placed in this position when the performance operator is used to develop an expression for the j^{th} output. The remaining terms identified input function product in the comparison variable position. This special case of subscript location is made in order to simplify the interpretations of the sensitivity ratio and the time response function ratio function. For example, in the simple situation in which the time response function is equal to unity, the specified input function product — reference input function product sensitivity ratio is equal to the ratio of the change in the reference input function product to the change in the specified input function product required to have equal effects on the selected output. This means that the significance of the basic subscript pattern is actually preserved by placing the reference input function product in the compared quantity position of the sensitivity ratio subscript because in this case the reference input function product is actually the compared quantity, with the specified input function product as the reference quantity so far as equivalent output effects are concerned. A line of reasoning similar to that for the sensitivity ratio can be used to justify the use of the reference input function product in the compared quantity position, and the specified input function product in the reference quantity position of the subscript of the time response function ratio function.

10.22. Details of the definitions and notation for sensitivity ratios and time response function ratio functions are given in Definition Summary 10-6.

SIMPLIFICATIONS IN PERFORMANCE FUNCTION NOTATION FOR PRACTICAL APPLICATIONS

10.23. Practical applications of performance functions are facilitated by simplifications that are possible in many situations. Certain of these simplifications are shown, step by step, in Definition Summary 10-7. The performance function symbol, with a full subscript system following the conventions established in Fig. 10-3, is very useful as a concise means for completely describing the behavior of an operating component. However, the complete symbols are too complex to be suitable for many applications. This complexity can usually be reduced by reasonable assumptions with regard to the physical situation represented by the symbol. Several circumstances that permit symbol simplifications are listed below:

> Modifying inputs with fixed values — In practice, the modifying inputs to an operating component are often set at fixed levels and, for this reason, are removed from the class of variables. The existence of constant modifying inputs is recognized by removing these inputs from the input-output subscript bracket of the performance function.
>
> Single input — For design, analysis, and specification purposes, it often happens that one input is of primary importance in the sense that all other inputs may be taken as constant or as having a negligible effect on the output under consideration if they are

GENERAL FORM **SPECIAL FORM**

$$[PF]_{(iden)(1),\ldots[q_{(in)(ac)(i)},\ldots,q_{(in)(ac)(k)};q_{(in)(mod)(l)},\ldots;q_{(out)(j)}](iden)(n)} \xrightarrow{\text{When all modifying inputs are constant (do not appear as variables in the situation considered.)}} [PF]_{(iden)(1),\ldots[q_{(in)(ac)(i)},\ldots q_{(in)(ac)(k)};q_{(out)(j)}](iden)(n)}$$

Note:
1. The performance function to the right of the arrow is in general different in dimensions from that to the left of the arrow.
2. Numerical values associated with the right-hand performance function depend upon the magnitudes used for the constant values of the modifying inputs.
3. In general, the omission of modifying inputs from performance function subscripts means that these inputs are not essential variables for which the operating component is considered.
4. Performance function symbols for interfering inputs, instead of actuating inputs, are formed by replacing subscript symbols (ac) by subscript symbols (int).

$$[PF]_{(iden)(1),\ldots[q_{(in)(ac)(i)},\ldots,q_{(in)(ac)(k)};q_{(out)(j)}](iden)(n)} \xrightarrow{\text{When only a single actuating input is important, or the operating component is sensitive to only a single variable.}} [PF]_{(iden)(1),\ldots[q_{(in)(ac)};q_{(out)}](iden)(n)}$$

Note: The use of a single input symbol in the subscript means that only one independent variable is important for the phase of operating component performance described by the given performance function.

$$[PF]_{(iden)(1),\ldots[q_{(in)(ac)};q_{(out)}](iden)(n)} \xrightarrow{\text{When a single subscript gives sufficient identification or when no misunderstanding of a more complicated situation is possible.}} [PF]_{(id)[q_{(in)(ac)};q_{(out)}]}$$

$$[PF]_{(id)[q_{(in)(ac)};q_{(out)}]} \xrightarrow{\text{When the only distinction between input and output quantities is important, without need for differentiating between actuating and interfering inputs.}} [PF]_{(id)[q_{(in)};q_{(out)}]} \xrightarrow{\text{When input and output quantities are replaced by reference and dependent quantities that are functions of the input quantity and output quantity respectively.}} [PF]_{(id)[q_{ref};q_{dep}]}$$

Note: The performance function for reference and dependent quantities is in general different from the input quantity—output quantity performance function.

$$[PF]_{(id)[q_{ref};q_{dep}]} \xrightarrow{\text{When the context insures that misunderstanding is not likely, the performance function symbol may be simplified.}} [PF]^{*}$$

Single Greek letter
Single English letter

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* Single letter or number subscripts as required may be added to simple symbols for performance functions.

Definition Summary 10-7. Simplified forms of performance function symbols.

not constant. In this situation, the distinction between input quantities is no longer necessary, and the symbols inside the input-output subscript bracket of the performance operator may be shortened to represent a single input and a single output.

Simplified operating component identification — When any reasonably extended discussion of the performance characteristics of a given operating component is to be carried out, it is ordinarily easy, without any possible misunderstanding, to reduce the number of subscripts necessary to identify the component. This is possible because all necessary identification may be made with a single subscript symbol. In Definition Summary 10-7, the complete identification group of subscript symbols is shown as reduced to one symbol, shortened to (id).

No distinction between actuating and interfering inputs – When the distinction between actuating input quantities and interfering input quantities is unimportant, the (ac) and (int) subscripts may be omitted from the input quantities symbols in the performance function subscript.

SUBSTITUTION OF REFERENCE AND DEPENDENT QUANTITIES FOR INPUT AND OUTPUT QUANTITIES

10.24. For many purposes, it is convenient to consider the performance of an operating component in terms of a reference quantity and a dependent quantity instead of inputs and outputs. These quantities are, in general, related to the inputs and outputs by the quantity transformations described in Definition Summary 10-8. This means that a performance function may be written with the reference quantity in the reference quantity subscript position and the dependent quantity in the compared quantity subscript position. As illustrated in the last symbol on the right-hand side of the lowest line of Definition Summary 10-7, the performance operator symbol may be reduced to the minimum complexity consistent with a reliable identification of the reference and dependent quantities and the operating component concerned.

PERFORMANCE FUNCTION REPRESENTATION

10.25. The ultimate simplification of notation for the representation of performance functions is to remove all subscripts from the main symbol. This procedure is not usable in complex situations where a number of different performance functions are involved, but simple number or letter subscripts will ordinarily supply all the necessary identification. It is also possible to represent performance functions by means of single Greek or English letters. Symbols of this type are particularly adapted for use as working variables during the formal design and analysis phases of instrument engineering work. Many examples of simple working variable symbols appear in Volume II.

QUANTITY TRANSFORMATIONS AND QUANTITY TRANSFORMATION RATIOS

10.26. It is so often desirable for instrument engineering problems to change the variables associated with the performance of operating components that it is worth while to develop concepts and notations for concisely describing transformations among physical quantities. For obvious reasons, changes of this kind are called quantity transformations, and are given the main symbol (QT). Quantity transformations involve relationships between physical quantities, so that to be self-defining their symbols must include subscript provisions for indicating the quantity "transformed from" and the quantity "transformed to." The quantity "transformed from" is, in effect, treated as the reference quantity, so that its symbol appears at the reference position in the quantity transformation subscript. The quantity "transformed to" corresponds to the compared quantity considered in the general scheme of subscript notation, and its symbol is given the compared quantity subscript position. In general, quantity transformations are based on natural laws and definitions, so that they are not directly associated with an operating component. This means that quantity transformation subscript systems do not have to include provisions for symbols to identify operating components. In special cases, the identification of quantity transformations may be required. Under these circumstances, additional subscripts may be added, when they become desirable, so as to satisfactorily identify some necessary relationship. Equation (1) and the definitions immediately below it in Definition Summary 10-8 define the quantity transformation in words and symbols. Equations (3) and (4)

By definition

$$q_{in} = (QT)_{[q_{ref};q_{in}]}q_{ref} \qquad q_{out} = (QT)_{[q_{dep};q_{out}]}q_{dep} \tag{1}$$

where

$(QT)_{[q_{ref};q_{in}]} \equiv$ Reference quantity – input quantity transformation

$(QT)_{[q_{dep};q_{out}]} \equiv$ Dependent quantity – output quantity transformation

In equation form

$$q_{out} = [PF]_{(id)[q_{in};q_{out}]}q_{in} \tag{2}$$

Using the quantity transformation definitions

$$q_{out} = q_{dep}(QT)_{[q_{dep};q_{out}]} = [PF]_{(id)[q_{in};q_{out}]}q_{in} = [PF]_{(id)[q_{in};q_{out}]}(QT)_{[q_{ref};q_{in}]}q_{ref} \tag{3}$$

so that

$$q_{dep} = [PF]_{(id)[q_{in};q_{out}]}(QTR)_{[q_{dep};q_{out}:q_{ref};q_{in}]}q_{ref} = [PF]_{(id)[q_{ref};q_{dep}]}q_{ref} \tag{4}$$

where by definition

$$(QTR)^{*}_{[q_{dep};q_{out}:q_{ref};q_{in}]} = \frac{(QT)_{[q_{ref};q_{in}]}}{(QT)_{[q_{dep};q_{out}]}} \equiv$$ Dependent quantity, output quantity – reference quantity, input quantity transformation ratio

Using this definition

$$[PF]_{(id)[q_{ref};q_{dep}]} = [PF]_{(id)[q_{in};q_{out}]}(QTR)_{[q_{dep};q_{out}:q_{ref};q_{in}]} \tag{5}$$

For the purposes of analysis and design it is convenient to use generalized reference and dependent quantities as the independent and dependent variables, respectively, in the form

$$q_{dep} = [PF]_{(id)[q_{ref};q_{dep}]}q_{ref} \tag{6}$$

---

* Quantity transformations may not always have the forms of scalar functions. If they involve functions of other types of quantity transformation ratios, the symbol (QTR) will become the symbol (QTRF), for quantity transformation ratio functions.

Definition Summary 10-8. Definitions and relationships for quantity transformations and changes from inputs and outputs to reference-quantity and dependent-quantity variables.

illustrate the use of the quantity transformation for changing from input and output quantities to reference and dependent quantities as variables. For purposes of making transformations of this type, it is convenient to define the quantity transformation ratio, with the main symbol (QTR). By definition, the quantity transformation ratio is the ratio of one quantity transformation to another. A pair of symbols representing the "transformed to" and the "transformed from" quantities for the quantity pair of the quantity transformation placed in the denominator of the quantity transformation ratio are separated by a semicolon and placed in the reference quantity subscript position. The corresponding pair of "transformed to" and "transformed from" quantity symbols for the quantity transformation used in the denominator of the quantity transformation ratio are placed in the compared quantity subscript position of the quantity transformation ratio. This second pair of symbols is separated from the first pair of symbols by a colon.

10.27. The particular usefulness of the quantity transformation is to formally provide for the replacement of input and output quantities in performance functions by reference and dependent quantities. Equation (5) of Definition Summary 10-8 shows that the reference quantity – dependent quantity performance function is equal to the input – output performance function multiplied by the dependent quantity, output quantity – reference quantity, input quantity transformation ratio. The final equation, (6), of Definition Summary 10-8 gives the formal expression for a dependent quantity in terms of a reference quantity and the reference quantity – dependent quantity performance function determined by the performance of an identified operating component. This form is very often useful as the basis for carrying through the theoretical reasoning necessary for analysis and design work in instrument engineering.

CHAPTER 11

APPLICATIONS OF STATISTICAL METHODS TO INSTRUMENT ENGINEERING

Uncertainty is one component of the difference between the actual level of a physical variable and the correct level. The other component of this difference is error, which is the difference between the averaged, or smoothed, level of the variable and the correct level as these concepts are defined in Chapter 9.

In general, error may be treated by the procedures of conventional analysis. On the other hand, uncertainty can be handled only in terms of probabilities by the methods of statistical analysis; i.e., it is not possible to predict exactly the uncertainty associated with a given indication of a physical quantity, but the probability that the uncertainty will fall within certain limits may be stated with some degree of confidence. It is the purpose of this section to give a condensed review of the parts of statistical analysis that are generally useful for instrument engineering and to provide diagrams and tables adapted for convenient use by engineers.

The material that follows is taken directly from a publication prepared for Instrumentation Laboratory, Massachusetts Institute of Technology, by H. B. Brainerd, who has given particular attention to the study of probability theory used in the Laboratory. Brainerd's material is presented here without modification.

PROBABILITY FOR INSTRUMENT ENGINEERING APPLICATIONS

H. B. Brainerd

FUNDAMENTAL DEFINITIONS

11.1. The mathematics of probability deals with uncertainties, also called random variables, or statistical quantities. The values of these quantities do not adhere individually to any systematic law, but en masse they do follow definite laws.

11.2. Let x = a variable which obeys the normal probability law. This law is a mathematical concept which idealizes the actual occurrence of many physical variables such as uncertainties in measurements. It is also called the Gaussian probability law.

11.3. The average value of x is defined as

$$(\mathrm{Av})x \equiv \frac{1}{n} \sum_{i=1}^{n} x_i \tag{11-1}$$

where x_i is a typical value of x, and n is the number of typical values considered.

11.4. The deviation of x, for each typical value, is defined as

$$(\mathrm{D})x \equiv x_i - (\mathrm{Av})x. \tag{11-2}$$

11.5. When x is considered to be made up of a constant correct value and an uncertainty

$$x = x_{(\mathrm{corr})} + (\mathrm{U})x, \tag{11-3}$$

(Av)x is the best estimate of $x_{(\mathrm{corr})}$ and (D)x is the best estimate of (U)x for each value of x_i.

11.6. The standard deviation of x is defined as the root mean square of the individual deviations of x:

$$(\mathrm{SD})x \equiv (\mathrm{RMS})\,[(\mathrm{D})x] \equiv \sqrt{(\mathrm{Av})\,[(\mathrm{D})x]^2} \tag{11-4}$$

11.7. The average deviation of x is defined as the average of the magnitudes of the individual deviations of x:

$$(\mathrm{AD})x \equiv (\mathrm{Av})\,|(\mathrm{D})x| \tag{11-5}$$

11.8. Either (SD)x or (AD)x is a convenient parameter for expressing the effective spread of a set of values about the average. The choice of which to use is largely a matter of convenience. If a slide-disk calculator is available, (SD)x is the easier to compute, but if the work is done arithmetically, (AD)x is the easier. If (AD)x is used, it is good practice to sum positive and negative deviations separately. Then the equality in the magnitude of the two sums, i.e., positive and negative deviations,

$$\left|\sum (D)x_{(pos)}\right| = \left|\sum (D)x_{(neg)}\right| \qquad (11\text{-}6)$$

checks that the arithmetic is correct to this point. If x obeys the normal probability law, it can be shown that (AD)x = 0.7979 (SD)x. In practice this relation will hold approximately rather than exactly.

11.9. The deviation – standard deviation ratio of x is defined as

$$(D(SD)R)x \equiv \frac{(D)x}{(SD)x} \qquad (11\text{-}7)$$

11.10. The deviation – average deviation ratio of x is defined as

$$(D(AD)R)x \equiv \frac{(D)x}{(AD)x} \qquad (11\text{-}8)$$

11.11. The use of either of these dimensionless ratios allows the behavior of any variable, x, which follows the normal probability law to be described by a dimensionless function and two parameters: (Av)x, which expresses the central value, and either (SD)x or (AD)x to express the range of deviations from this central value.

11.12. The probability of any specified event is defined as a number between zero (i.e., impossibility) and unity (i.e., certainty) which expresses the expectation that this event will occur. This definition implies the existence of theoretical or empirical evidence permitting assignment of a numerical value to the probability, and a defined range of possibilities which includes the specified event.

11.13. The numerical value of a probability expresses the expected ratio of specified events to total possibilities when a very large number of trials are made. Two events which are equally likely have equal probabilities. The probability of one of several mutually exclusive events is the sum of the probabilities of these events. Represent the probability of a specified event with the symbol

$$(P) \quad \{\text{specified event}\}$$

11.14. To apply probability to the numerical values of a variable, consider a variable, x, which may fall anywhere within the limits $x_{(lim)0}$ to $x_{(lim)n}$. Assume that this range can be divided into n increments ($x_{(\mathrm{lim})0}$ to $x_{(\mathrm{lim})1}, \ldots x_{(\mathrm{lim})(n-1)}$ to $x_{(\mathrm{lim})n}$) which may differ in size but which have the property that a typical value of x is equally likely to fall in any one of the increments. Then, since the range from $x_{(\mathrm{lim})0}$ to $x_{(\mathrm{lim})n}$ covers all possible values, it is certain that a typical value of x will fall between them, i.e.,

$$(P)\left\{x_{(\mathrm{lim})0} < x_i < x_{(\mathrm{lim})n}\right\} = 1. \tag{11-9}$$

Since the n increments add up to the whole range, and since it is equally likely that x will fall in any one of them, the probability of x_i falling in the k^{th} increment is 1/n, i.e.,

$$(P)\left\{x_{(\mathrm{lim})(k-1)} < x_i < x_{(\mathrm{lim})k}\right\} = \frac{1}{n} \tag{11-10}$$

and for a sequence of d increments, the probability is d/n, i.e.,

$$(P)\left\{x_{(\mathrm{lim})(k-1)} < x_i < x_{(\mathrm{lim})(k+d)}\right\} = \frac{d}{n}. \tag{11-11}$$

The conventional methods of the calculus may be used to extend this treatment to the limiting case of an infinite range of values and infinitesimal increments.

11.15. At this point it is convenient to introduce two more parameters, either of which may be used in place of the standard deviation or the average deviation.

11.16. The probable deviation of x is defined as that deviation for which the probability of a typical value having a smaller deviation is one half, i.e.,

$$(P)\left\{\left|(D)x\right| < \left|(PD)x\right|\right\} \equiv \frac{1}{2}. \tag{11-12}$$

In other words, (PD)x is the magnitude such that one-half the individual deviations can be expected to be smaller, and the other half larger, than (PD)x. If x obeys the normal probability law, it can be shown that (PD)x = 0.6745 (SD)x. In practical cases this ratio will hold approximately rather than exactly.

11.17. The deviation — probable deviation ratio of x is defined as

$$(D(PD)R)x \equiv \frac{(D)x}{(PD)x} \tag{11-13}$$

11.18. The modulus of precision of x is defined as

$$(MP)x \equiv \frac{1}{(SD)x\sqrt{2}}. \tag{11-14}$$

Example of calculation of averages and deviation parameters:

Twelve vacuum tubes are tested under identical conditions, and the plate current of each tube is measured. Compute the following for the plate current, x, as given below: average, average deviation, standard deviation, probable deviation, modulus of precision, deviation – average deviation ratio, deviation – standard deviation ratio. (The data have been rearranged in order of size for convenience in computing.)

x (ma)	(D)x (ma) +	(D)x (ma) -	$[(D)x]^2$ $(ma)^2$	(D(AD)R)x (dimensionless)	(D(SD)R)x (dimensionless)
37.2		- 1.75	3.0625	- 2.19	- 1.77
37.6		- 1.35	1.8225	- 1.69	- 1.36
38.2		- 0.75	0.5625	- 0.94	- 0.76
38.5		- 0.45	0.2025	- 0.56	- 0.45
38.6		- 0.35	0.1225	- 0.44	- 0.35
38.8		- 0.15	0.0225	- 0.19	- 0.15
39.0	+ 0.05		0.0025	+ 0.06	+ 0.05
39.1	+ 0.15		0.0225	+ 0.19	+ 0.15
39.5	+ 0.55		0.3025	+ 0.69	+ 0.56
40.0	+ 1.05		1.1025	+ 1.31	+ 1.06
40.3	+ 1.35		1.8225	+ 1.69	+ 1.36
40.6	+ 1.65		2.7225	+ 2.06	+ 1.67
12)467.4	4.80	4.80	12)11.7700		
38.95 ma			0.98083		
= (Av)x					

12)9.60

0.80 ma = (AD)x

$$(SD)x = \sqrt{0.98083} = 0.99 \text{ ma}$$

$$(MP)x = \frac{1}{\sqrt{2}\ (SD)x} = 0.713 \text{ ma}^{-1}$$

Since six deviations range from 0.05 ma to 0.55 ma, and the other six range from 0.75 ma to 1.75 ma, (PD)x lies between 0.55 ma and 0.75 ma. The value of 0.6745 (SD)x called for by the normal probability law, 0.67 ma, lies near the middle of this range.

Note that, in general, averages and deviation parameters should be computed to one more decimal place than the original data.

11.19. The *normal probability frequency function of x* is defined as

$$(\text{NPFF})x = \frac{1}{(\text{SD})x\sqrt{2\pi}}\, e^{-\frac{1}{2}[(\text{D}(\text{SD})\text{R})x]^2} \tag{11-15}$$

This function (often called 'the bell-shaped curve') is illustrated in Fig. 11-1. It describes the relative frequency of occurrence or "probability density" of any variable which follows the normal probability law.

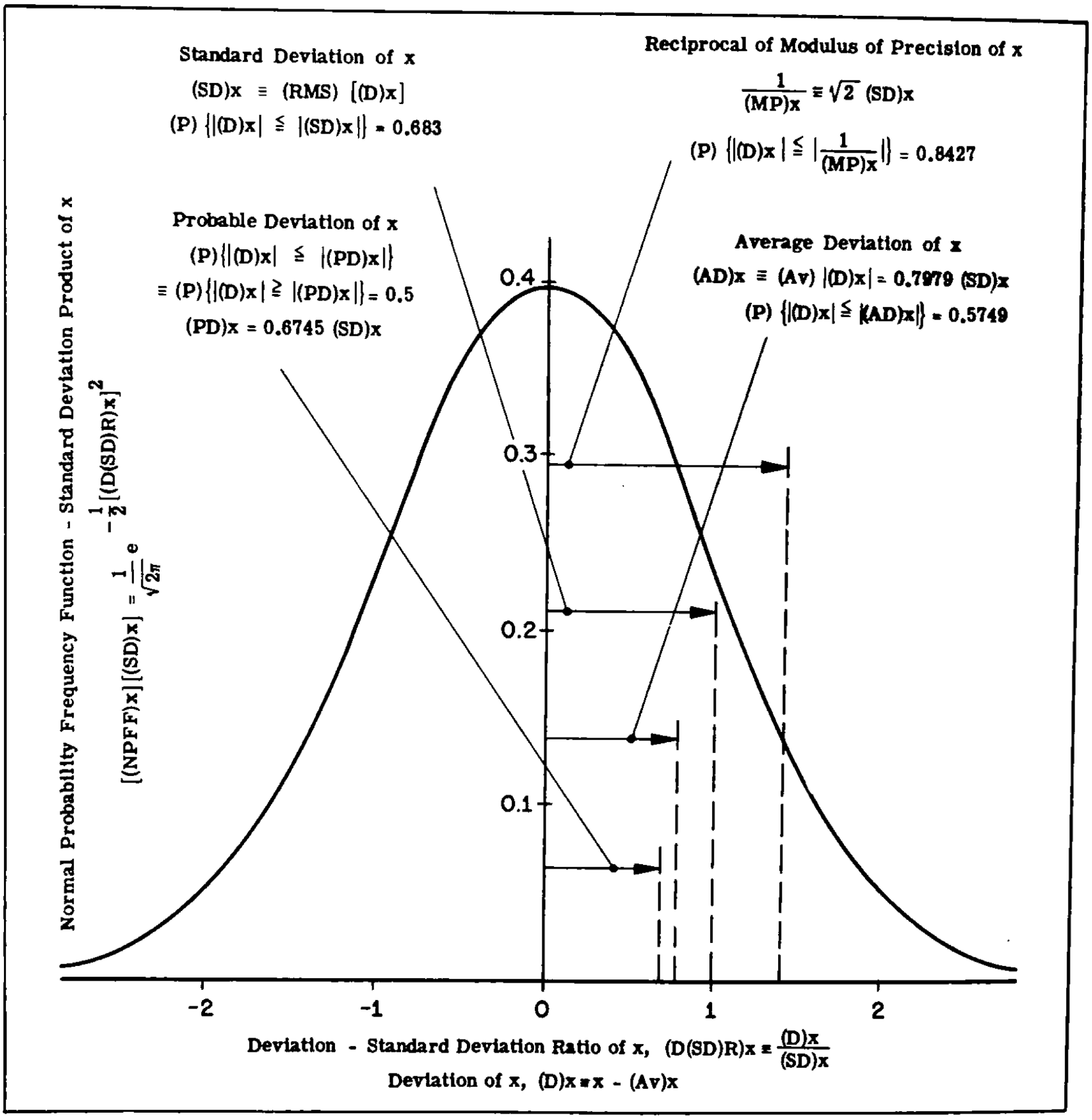
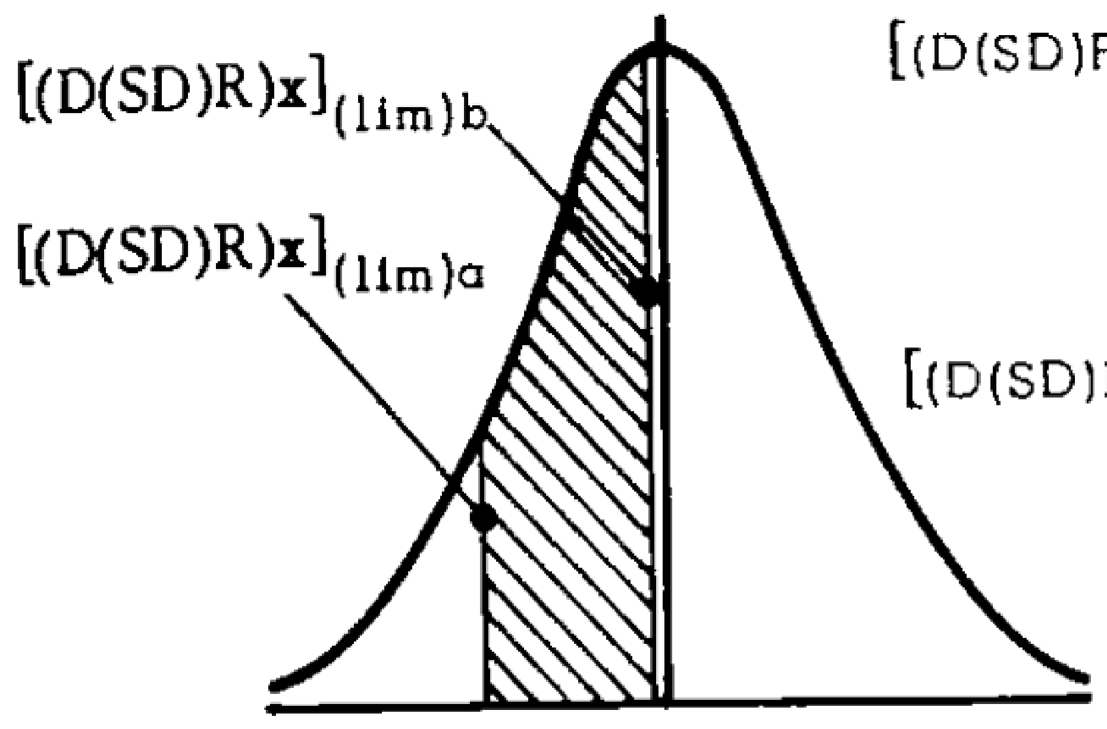


Fig. 11-1. The normal probability frequency function of x.

11.20. The normal probability frequency function has the following useful properties:

(1) The area under the (NPFF)x curve between any two limits is the probability that a typical value of (D(SD)R)x will fall within these limits, i.e.,



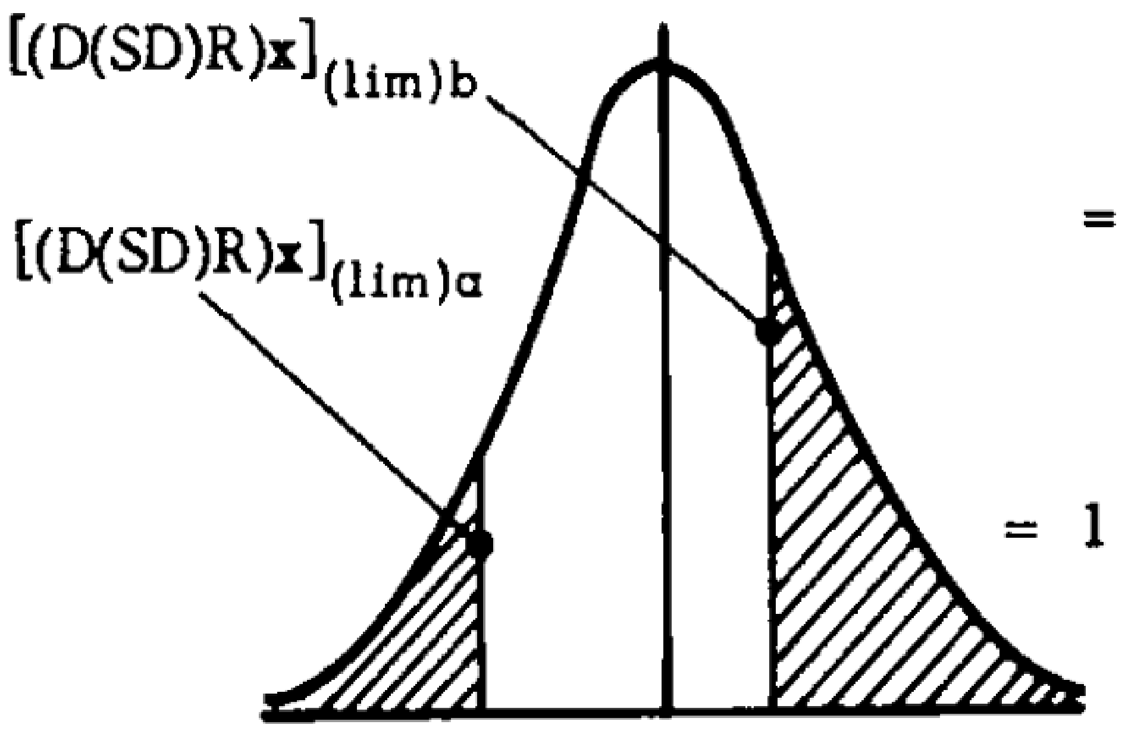
$$\int_{[(D(SD)R)x]_{(lim)a}}^{[(D(SD)R)x]_{(lim)b}} [(NPFF)x]d[(D(SD)R)x]$$

$$= (P)\{[(D(SD)R)x]_{(lim)a} < (D(SD)R)x < [(D(SD)R)x]_{(lim)b}\} \qquad (11\text{-}16)$$

(2) The total area under the (NPFF)x curve is unity, and the probability of a typical value of (D(SD)R)x falling outside any limits is one minus the probability of its falling inside the same limits,

$$\int_{-\infty}^{[(D(SD)R)x]_{(lim)a}} [(NPFF)x]d[(D(SD)R)x] + \int_{[(D(SD)R)x]_{(lim)b}}^{\infty} [(NPFF)x]d[(D(SD)R)x]$$

$$= 1 - \int_{[(D(SD)R)x]_{(lim)a}}^{[(D(SD)R)x]_{(lim)b}} [(NPFF)x]d[(D(SD)R)x] \qquad (11\text{-}17a)$$

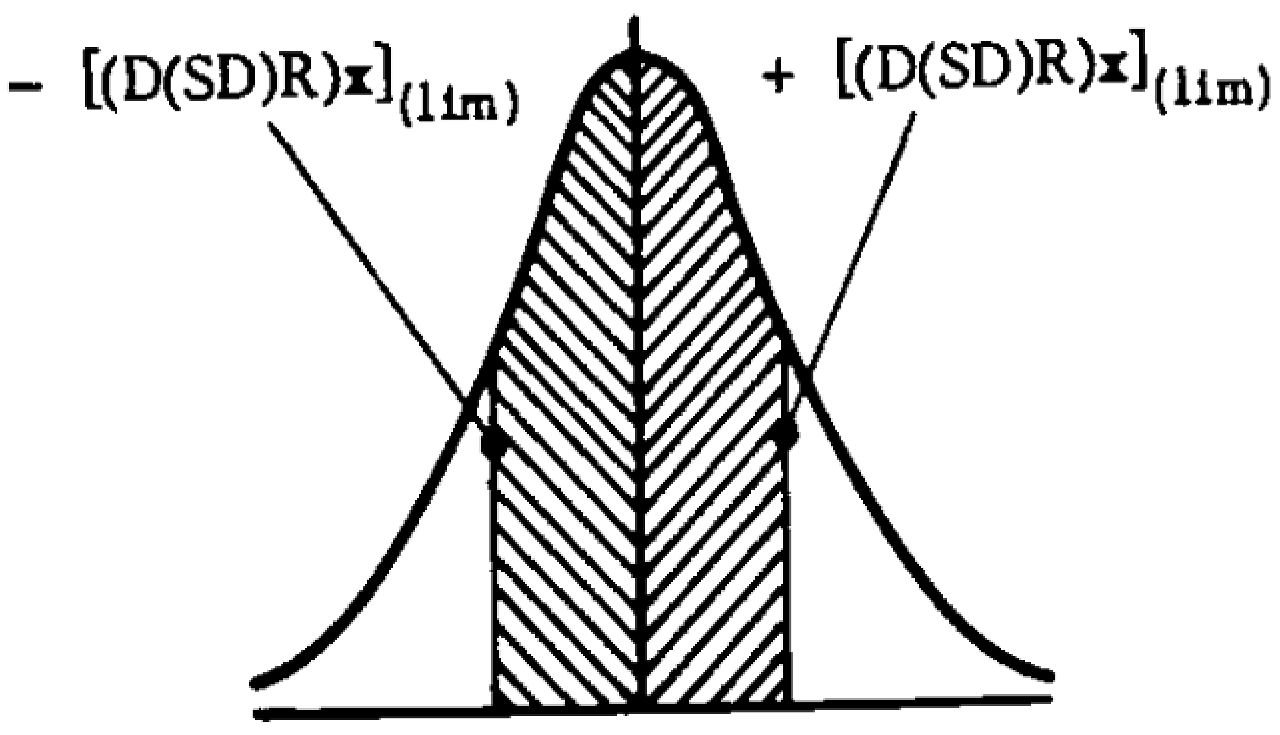


$$= (P)\left\{\begin{array}{c} (D(SD)R)x < [(D(SD)R)x]_{(lim)a} \\ \text{or} \\ (D(SD)R)x > [(D(SD)R)x]_{(lim)b} \end{array}\right\} \qquad (11\text{-}17b)$$

$$= 1 - (P)\{[(D(SD)R)x]_{(lim)a} < (D(SD)R)x < [(D(SD)R)x]_{(lim)b}\} \qquad (11\text{-}17c)$$

(3) The (NPFF)x curve is symmetrical, so that equal positive and negative deviations enclose equal areas with equal probabilities,

$$\int_{-[(D(SD)R)x]_{(lim)}}^{0} [(NPFF)x]d[(D(SD)R)x] = \int_{0}^{+[(D(SD)R)x]_{(lim)}} [(NPFF)x]d[(D(SD)R)x] \qquad (11\text{-}18a)$$

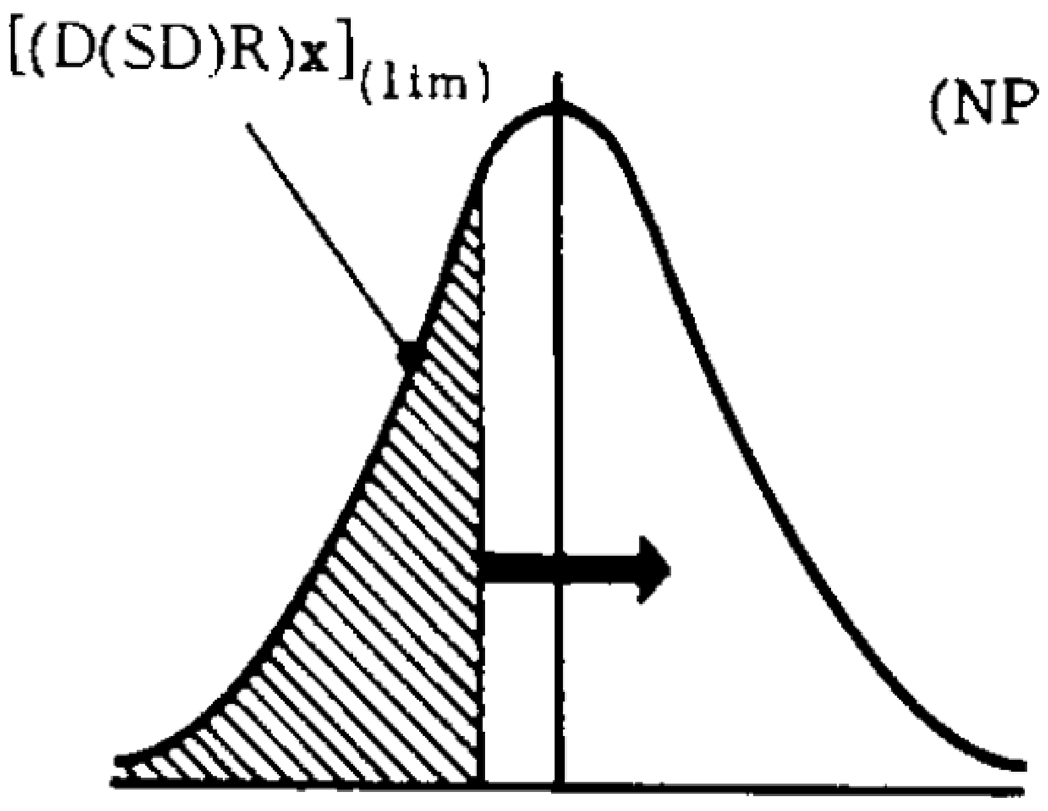


$$= (P)\{-[(D(SD)R)x]_{(lim)} < (D(SD)R)x < 0\} \qquad (11\text{-}18b)$$

$$= (P)\{0 < (D(SD)R)x < +[(D(SD)R)x]_{(lim)}\} \qquad (11\text{-}18c)$$

11.21. Two choices of limits for the integral of (NPFF)x are important enough to be given specific names: the normal probability distribution function, and the normal magnitude probability function.

11.22. The normal probability distribution function of x is defined as the probability that a typical value of (D(SD)R)x will lie between minus infinity and $[(D(SD)R)x]_{(lim)}$.



$$(NPDF)x \equiv (P)\left\{-\infty < (D(SD)R)x < [(D(SD)R)x]_{(lim)}\right\} \qquad (11\text{-}19a)$$

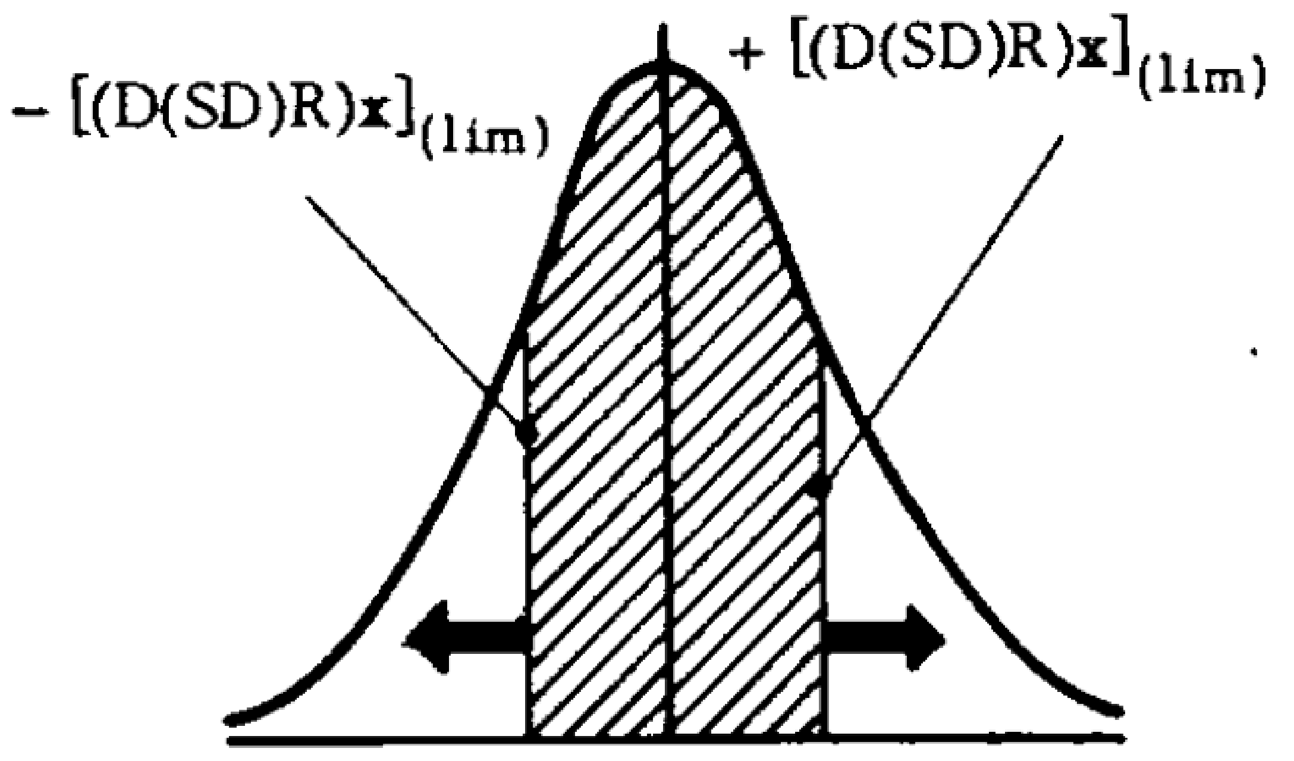
$$= \int_{-\infty}^{[(D(SD)R)x]_{(lim)}} [(NPFF)x]d[(D(SD)R)x] \qquad (11\text{-}19b)$$

Note that (NPDF)x is a function of $[(D(SD)R)x]_{(lim)}$ as this limit is allowed to travel from $-\infty$ to $+\infty$. (NPDF)x can also be expressed as a function of $[(D(AD)R)x]_{(lim)}$, $[(D(PD)R)x]_{(lim)}$, or $[(D)x(MP)x]_{(lim)}$. (NPDF)x is illustrated in Figs. 11-2 and 11-4, and tabulated in Tables 11-1, 11-2, 11-3, and 11-4.

11.23. The normal magnitude probability function of x is defined as the probability that (D(SD)R)x will be less in magnitude than $[(D(SD)R)x]_{(lim)}$, which is equivalent to (D(SD)R)x lying between $-[(D(SD)R)x]_{(lim)}$ and $+[(D(SD)R)x]_{(lim)}$.

$$(NMPF)x \equiv (P)\left\{|(D(SD)R)x| < |[(D(SD)R)x]_{(lim)}|\right\} \qquad (11\text{-}20a)$$

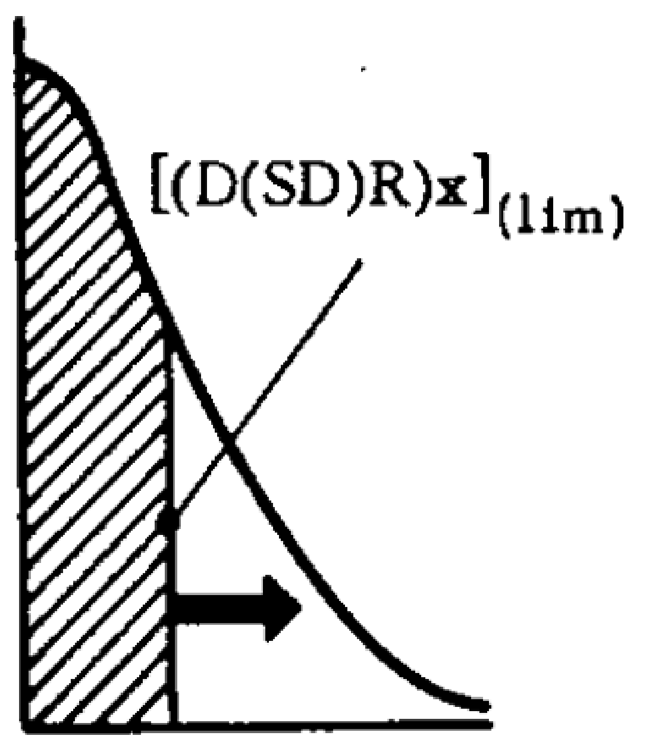
$$= (P)\left\{-[(D(SD)R)x]_{(lim)} < (D(SD)R)x < +[(D(SD)R)x]_{(lim)}\right\} \qquad (11\text{-}20b)$$



$$= \int_{-[(D(SD)R)x]_{(lim)}}^{+[(D(SD)R)x]_{(lim)}} [(NPFF)x]d[(D(SD)R)x] \qquad (11\text{-}20c)$$

$$= 2\int_{0}^{[(D(SD)R)x]_{(lim)}} [(NPFF)x]d[(D(SD)R)x] \qquad (11\text{-}20d)$$

or 2 ×

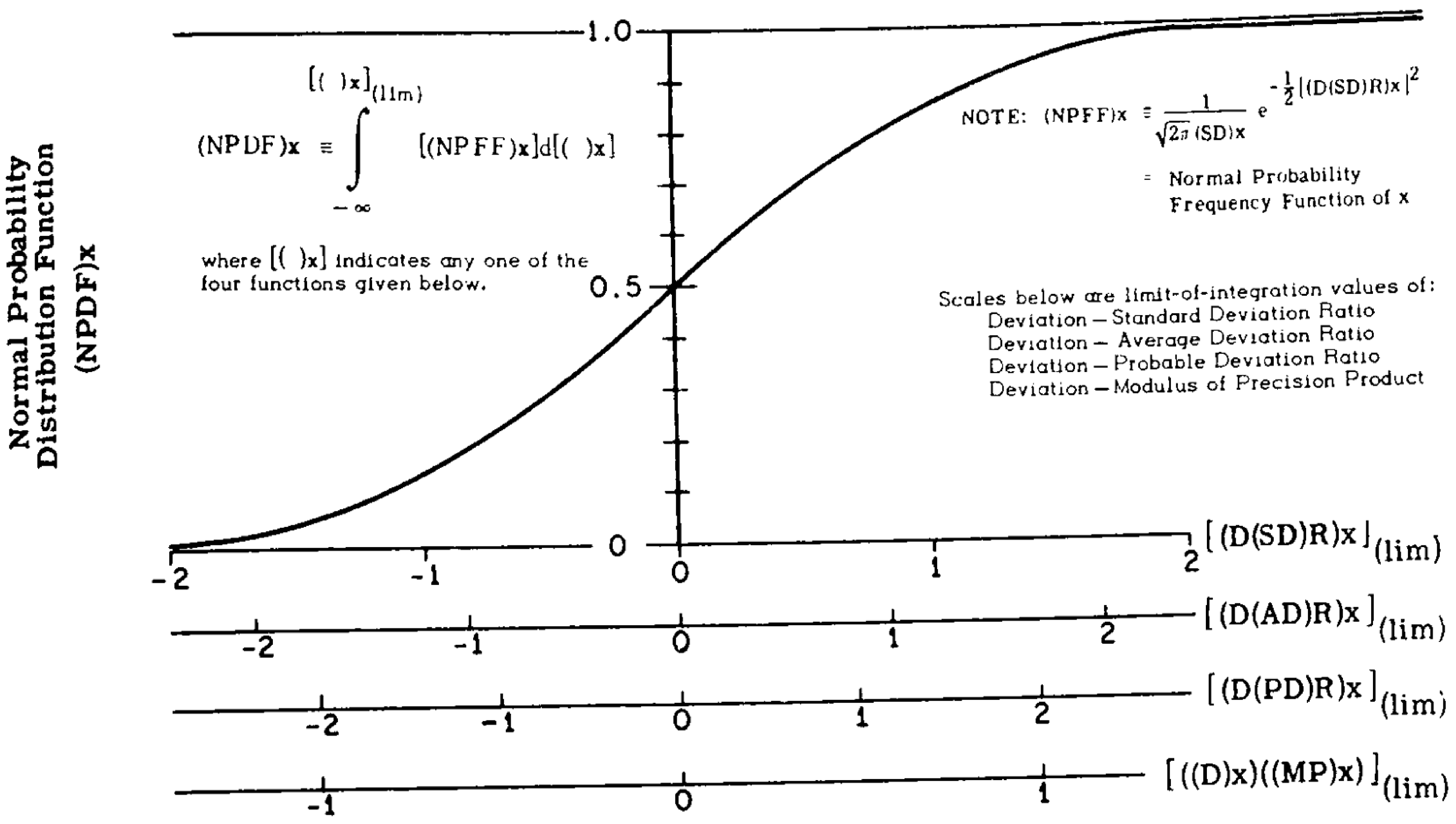


Note that (NMPF)x is a function of $[(D(SD)R)x]_{(lim)}$ as this limit is allowed to travel from zero to infinity. (NMPF)x can also be expressed as a function of $[(D(AD)R)x]_{(lim)}$, $[(D(PD)R)x]_{(lim)}$, or $[(D)x(MP)x]_{(lim)}$. (NMPF)x is illustrated in Figs. 11-3 and 11-4, and is tabulated in Tables 11-1, 11-2, 11-3, and 11-4.

11.24. It can be shown that

$$(NPDF)x = \frac{1}{2}[1 \pm (NMPF)x] \qquad (11\text{-}21)$$

where + or - is chosen to correspond to positive or negative deviations of x.



$$(\mathrm{NPDF})x \equiv \int_{-\infty}^{[(\ )x]_{(\mathrm{lim})}} [(\mathrm{NPFF})x]\,d[(\ )x]$$

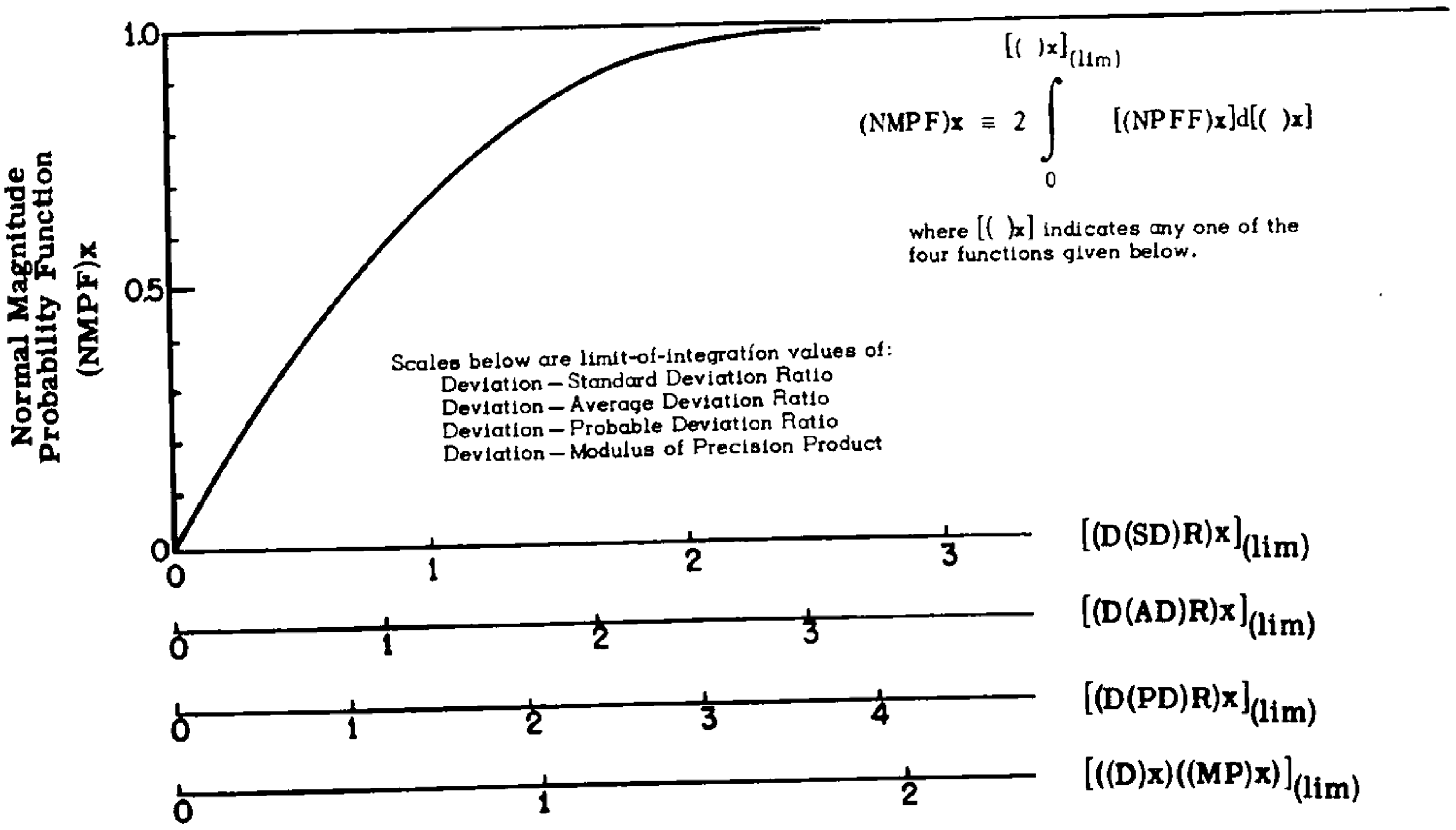
where [( )x] indicates any one of the four functions given below.

NOTE: $(\mathrm{NPFF})x \equiv \frac{1}{\sqrt{2\pi}\,(\mathrm{SD})x}\, e^{-\frac{1}{2}[(D(SD)R)x]^2}$

= Normal Probability Frequency Function of x

Scales below are limit-of-integration values of:
Deviation – Standard Deviation Ratio
Deviation – Average Deviation Ratio
Deviation – Probable Deviation Ratio
Deviation – Modulus of Precision Product

Fig. 11-2. The normal probability distribution function of x.



$$(\mathrm{NMPF})x \equiv 2\int_{0}^{[(\ )x]_{(\mathrm{lim})}} [(\mathrm{NPFF})x]\,d[(\ )x]$$

where [( )x] indicates any one of the four functions given below.

Scales below are limit-of-integration values of:
Deviation – Standard Deviation Ratio
Deviation – Average Deviation Ratio
Deviation – Probable Deviation Ratio
Deviation – Modulus of Precision Product

Fig. 11-3. The normal magnitude probability function of x.

DEFINITIONS OF DEVIATION

Let $x \equiv$ a variable which obeys the normal probability law. Assume that n typical values of x are known.

$$(Av)x \equiv (1/n) \sum x ;$$

the average of x is the sum of the values of x divided by the number of values.

$$(D)x \equiv x - (Av)x ;$$

the deviation of each value of x is that value minus the average of x.

$$(SD)x \equiv \sqrt{(Av)[(D)x]^2} ;$$

the standard deviation of x is the square root of the average of the squares of the individual deviations of x.

$$(AD)x \equiv (Av) \left|(D)x\right| ;$$

the average deviation of x is the average of the magnitudes of the individual deviations of x.

$$(P)\{|(D)x| < |(PD)x|\} = .5 ;$$

the probable deviation of x is that deviation for which there is a probability of one-half that an individual deviation of x is less in magnitude.

$$(D(SD)R)x \equiv (D)x/(SD)x ;$$

the deviation-standard deviation ratio of x is the ratio of an individual deviation of x to the standard deviation of x. Likewise:

$$(D(AD)R)x \equiv (D)x/(AD)x$$

$\equiv$ the deviation-average deviation ratio of x;

$$(D(PD)R)x \equiv (D)x/(PD)x$$

$\equiv$ the deviation-probable deviation ratio of x.

EXAMPLE

If a sample of steel balls from a production run have an average diameter of .25050 inch, and an average deviation of .00050 inch, what percentage can be expected to lie within .25000 ± .00100 inch? Let x = diameter.

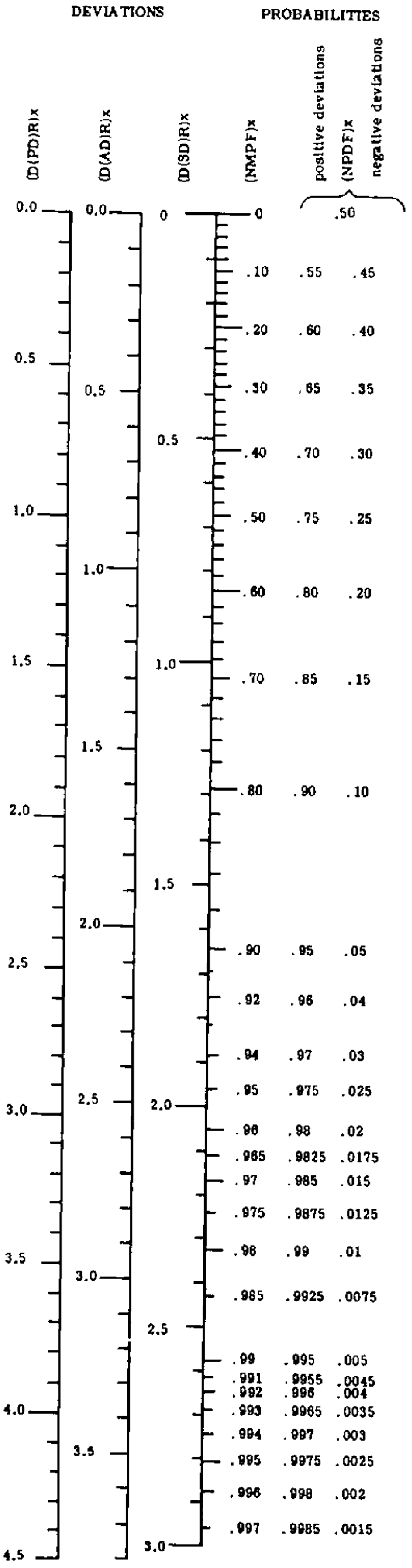
$(Av)x = .25050 \quad (AD)x = 0.00050$

$(D).25100 = +0.00050$
$(D(AD)R).25100 = +1.0;$
$(NPDF).25'00 = 0.79;$

i.e., 79% should be less than 0.25100;

$(D).24900 = -0.00150;$
$(D(AD)R).24900 = -3.0;$
$(NPDF).24900 = 0.01;$

i.e., 1% should be less than .24900; 79% − 1% = 78% of the balls should be between 0.24900 and 0.25100 inch in diameter.



DEFINITIONS OF PROBABILITIES

$$(NMPF)x \equiv (P)\{|(D)x| < |(D)x|_{(lim)}\} ;$$

the normal magnitude probability function of x is the probability that the magnitude of the deviation of x is less than the limit-of-integration value, i.e., the scale value of (D(SD)R)x, etc.

$$(NPDF)x \equiv (P)\{(D)x < (D)x_{(lim)}\} ;$$

the normal probability distribution function of x is the probability that the deviation of x lies between minus infinity and the limit of integration value, i.e., the scale value of (D(SD)R)x, etc.

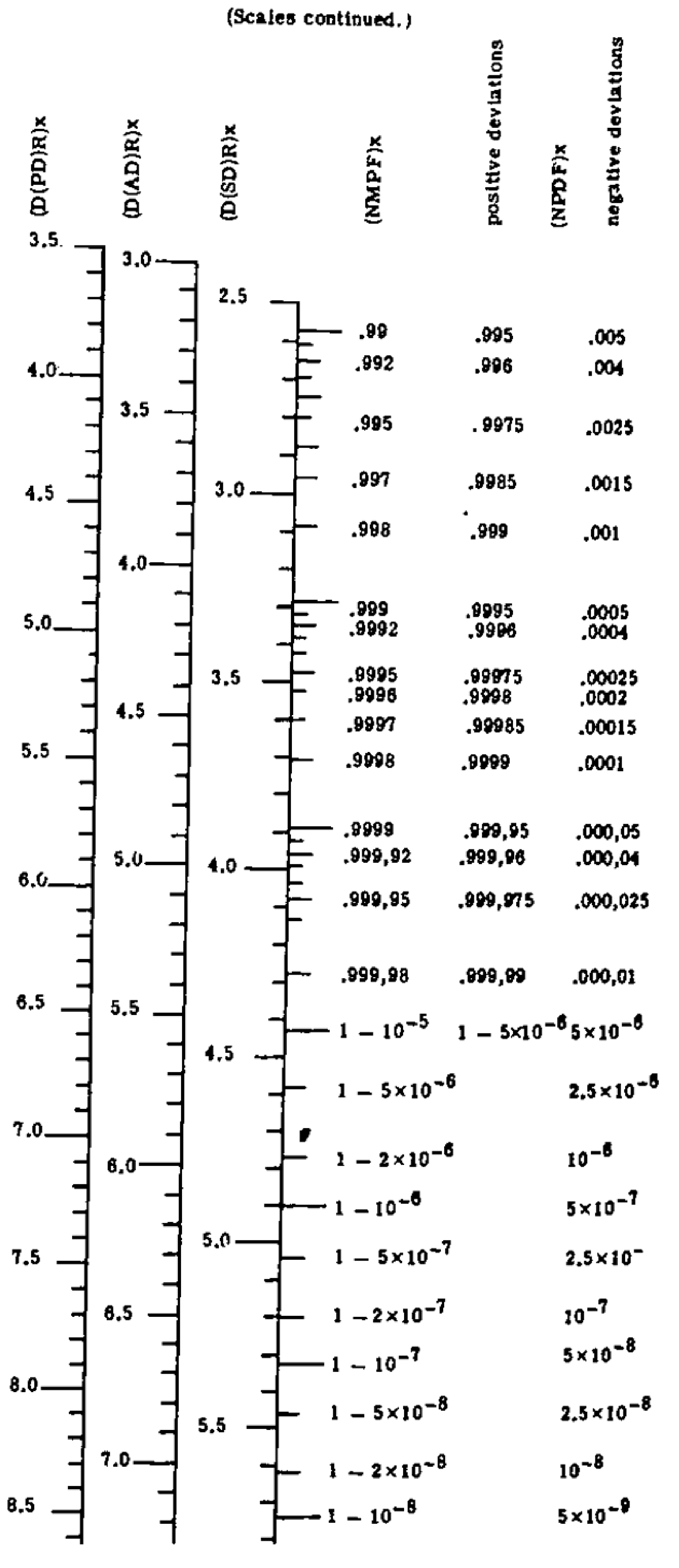


Fig. 11-4. Scale of probability values for various deviation ratios.

UNCERTAINTY IN A SET OF n VALUES

11.25. When the normal probability law is applied to an engineering problem, the available information is usually a set of n values of some variable, x, where n is a comparatively small number. The average of a finite number of typical values of x, $[(Av)x]_n$, is defined as the value of (Av)x which is computed from a set of n typical values. The average of an infinite number of typical values of x, $[(Av)x]_\infty$, is defined as the value of (Av)x which would be obtained from an infinite number of typical values.

11.26. It is evident that since x contains an uncertainty, different sets of n typical values can be expected to yield different values $[(Av)x]_n$. It can be shown that if a large number of such sets were available, the values of $[(Av)x]_n$ obtained from them would obey the normal probability law. The average of these averages would be the average of an infinite number of typical values of x:

$$(Av)[(Av)x]_n = [(Av)x]_\infty \tag{11-22}$$

The standard deviation of these average values from their collected average would be approximately $1/\sqrt{n}$ times the standard deviation of the values in a set from their average:

$$(SD)[(Av)x]_n = \frac{(SD)x}{\sqrt{n}} \tag{11-23}$$

11.27. In most practical cases, only one set of n values is available, and the problem is to estimate the magnitude of uncertainty, which is the difference between the computed average of the set and the unknown average of an infinite number of values; i.e., to estimate

$$|(U)[(Av)x]_n| \equiv |(D)[(Av)x]_n| \quad , \tag{11-24}$$

where

$$(D)[(Av)x]_n \equiv [(Av)x]_n - [(Av)x]_\infty \tag{11-25}$$

Since $[(Av)x]_n$ follows the normal probability law, there is no fixed limit on its deviations. Any deviation, however large, has some very small probability of occurring. But a limit can be set by specifying a confidence figure, i.e., a probability approaching unity. For example, specifying a confidence figure of 0.95 sets an uncertainty limit of $2(SD)[(Av)x]_n$, which is $[2(SD)x]/\sqrt{n}$ or $[2\text{-}1/2\ (AD)x]/\sqrt{n}$. This is another way of saying that there is 0.95 probability that $(U)[(Av)x]_n < [2(SD)x]/\sqrt{n}$ or $[2\text{-}1/2\ (AD)x]/\sqrt{n}$. Likewise, a confidence figure of 0.99-3/4 sets an uncertainty limit of $[3(SD)x]/\sqrt{n}$ or $[3\text{-}3/4\ (AD)x]/\sqrt{n}$.

11.28. The standard deviation (or other deviation parameter) of a set of n typical values of x also contains a relatively small uncertainty which was neglected in the foregoing discussion. The standard deviation of a set of n values of x is defined as that value which is computed from the set

$$[(SD)x]_n \equiv (RMS)\left[(x_i)\Big|_1^n - [(Av)x]_n\right] \qquad (11\text{-}26)$$

where $(x_i)\Big|_1^n$ represents each typical value in turn, in a set of n values. The standard deviation of an infinite number of values of x is defined as that value which would be computed from the deviation of an infinite number of values, if they were available

$$[(SD)x]_\infty \equiv (RMS)\left[(x_i)\Big|_1^\infty - [(Av)x]_\infty\right] \qquad (11\text{-}27)$$

11.29. It can be shown that the square of the standard deviation follows the normal probability law, with the following parameters

$$(Av)\,[(SD)x]_n^2 = [(SD)x]_\infty^2 \qquad (11\text{-}28)$$

$$(SD)\,[(SD)x]_n^2 = \frac{1}{n-1}\,[(SD)x]_\infty^2 \qquad (11\text{-}29)$$

11.30. The limits for the uncertainty in $[(SD)x]_n$, for a specified confidence figure, can be conveniently represented by an uncertainty standard deviation ratio, such that multiplying the computed standard deviation by this ratio will give the corresponding uncertainty limit, i.e.,

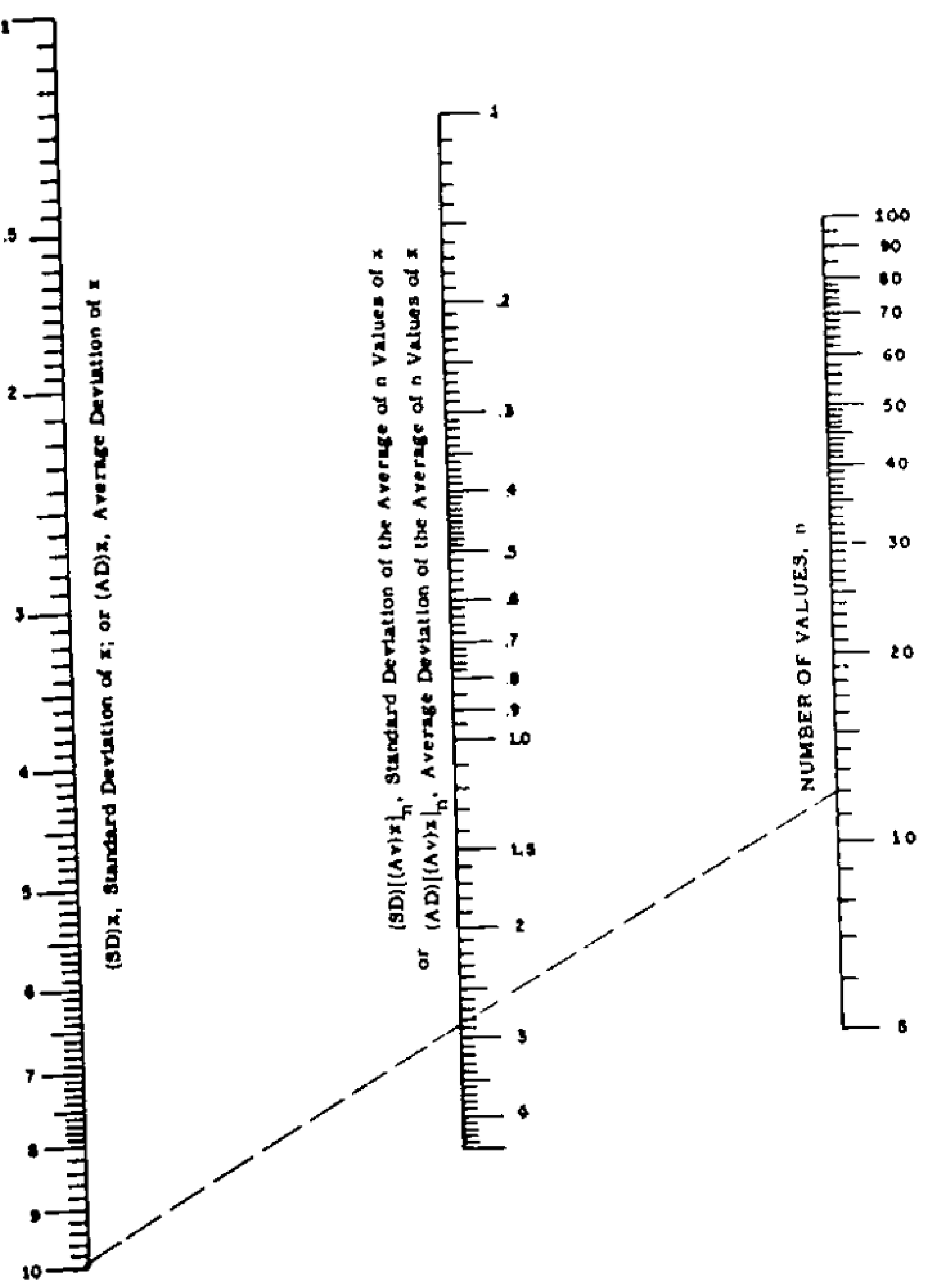
$$[(SD)x]_n\,(U(SD)R)\,[n, CF] = (UL)[(SD)x]_{(n,\ CF)} \qquad (11\text{-}30)$$

where $(U(SD)R)\,[n, CF]$ is the uncertainty standard deviation ratio, and $(UL)\,[(SD)x]_{(n,\ CF)}$ is the corresponding uncertainty limit. This uncertainty factor will have two values corresponding to the upper and lower uncertainty limits. Scales for computing $(U(SD)R)\,[n, CF]$ are given in Fig. 11-6.

11.31. It should be noted that some authorities give

$$(SD)\,[(Av)x]_n = \frac{1}{\sqrt{n-1}}\,(SD)x. \qquad (11\text{-}31)$$

This use of $\sqrt{n-1}$ rather than $\sqrt{n}$ has the effect of making an allowance for the uncertainty in $[(SD)x]_n$.


1
1.5
2
3
4
5
6
7
8
9
10
(SD)x, Standard Deviation of x; or (AD)x, Average Deviation of x
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
1.5
2
3
4
(SD)[(Av)x]n, Standard Deviation of the Average of n Values of x
or (AD)[(Av)x]n, Average Deviation of the Average of n Values of x
100
90
80
70
60
50
40
30
20
10
8
NUMBER OF VALUES, n

$$(SD)[(Av)x]_n = \frac{(SD)x}{\sqrt{n}}$$

$$(AD)[(Av)x]_n = \frac{(AD)x}{\sqrt{n}}$$

Example: If the plate current of twelve vacuum tubes averages 38.95 milliamperes and has a standard deviation of 0.99 milliampere, what is the probability that the average plate current of a large number of similar tubes will lie between 38.5 and 39.4 milliamperes (i.e., within 38.95 + 0.45 milliamperes)?

From this chart, when n = 12, and (SD)x = 9.9, $(SD)[(Av)x]_n$ = 2.85. Shifting the decimal point to 0.99 gives $(SD)[(Av)x]_n$ = 0.285.

Then when $(D)[(Av)x]_n$ = 0.45,

$$(D(SD)R)[(Av)x]_n = \frac{0.45}{0.285} = 1.58$$

Then from Fig. 11-4 $(NMPF)[(Av)x]_n$ = 0.885, which is the probability that the average of a large number of tubes will lie within the given limits.

Note: Moving the decimal point in (SD)x or (AD)x moves it the same number of places in $(SD)[(Av)x]_n$ or $(Ad)[(Av)x]_n$.

Fig. 11-5. Alignment chart for computing the standard deviation, or the average deviation, of the average of n values of x.

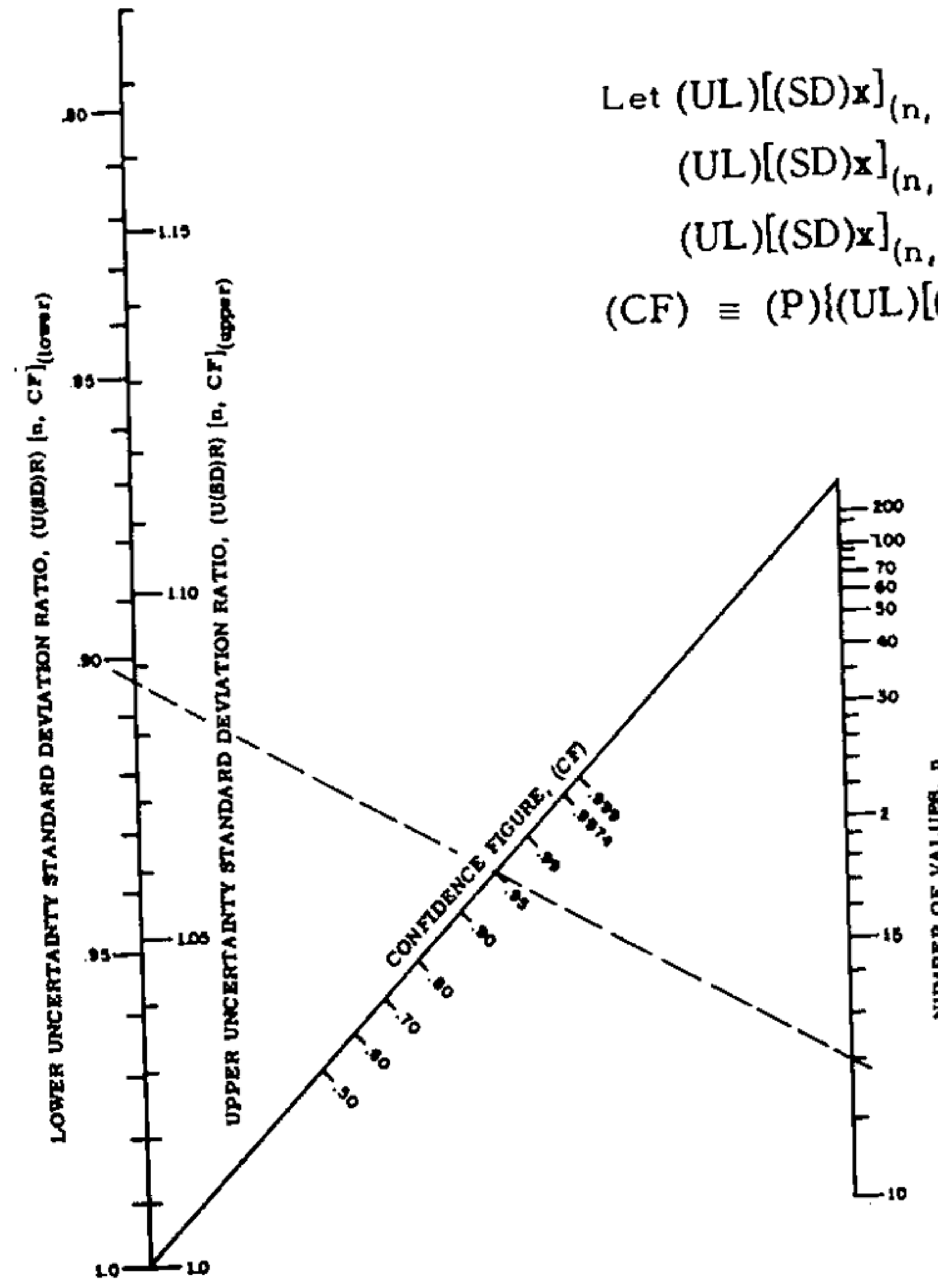
Definitions

Let $(UL)[(SD)x]_{(n,\ CF)} \equiv$ Uncertainty limit for the standard deviation of x.

$$(UL)[(SD)x]_{(n,\ CF)(lower)} \equiv [(SD)x]_n (U(SD)R)[n, CF]_{(lower)}$$

$$(UL)[(SD)x]_{(n,\ CF)(upper)} \equiv [(SD)x]_n (U(SD)R)[n, CF]_{(upper)}$$

$$(CF) \equiv (P)\{(UL)[(SD)x]_{(n,\ CF)(lower)} < [(SD)x]_\infty < (UL)[(SD)x]_{(n,\ CF)(upper)}\}$$


.80
.85
.90
.95
1.0
1.15
1.10
1.05
1.0
LOWER UNCERTAINTY STANDARD DEVIATION RATIO, (U(SD)R) [n, CF](lower)
UPPER UNCERTAINTY STANDARD DEVIATION RATIO, (U(SD)R) [n, CF](upper)
CONFIDENCE FIGURE, (CF)
.999
.9974
.99
.95
.90
.80
.70
.60
.50
200
100
70
60
50
40
30
2
15
10
NUMBER OF VALUES, n

Example: Find the uncertainty limits for a confidence figure of 0.95, when the standard deviation of twelve electric current readings is 0.99 milliampere:

The lower and upper values of (U(SD)R) [n CF] determined by a line through n = 12 and (CF) = 0.95 are 0.904 and 1.087 respectively (see dashed line). Multiplying 0.990 by these ratios gives limits of 0.895 and 1.077 milliamperes for the standard deviation of a very large number of readings.

Fig. 11-6. Alignment chart for computing the uncertainty in the standard deviation of n values of a variable x.

Table 11-1
Limiting values of the various deviation ratios for given values of the normal probability functions

(NPDF)x for negative deviations	(NPDF)x for positive deviations	1 − (NMPF)x	(NMPF)x	$[(D(SD)R)x]_{(lim)}$	$[(D(AD)R)x]_{(lim)}$	$[(D(PD)R)x]_{(lim)}$
		Probability that (D)x will be greater	less			
Probability that x will lie between minus infinity and		in magnitude than the limit shown, i.e., that x will lie		Limit expressed as Deviation-		
minus	plus	outside	inside	Standard	Average	Probable
the limit shown.		the range between minus and plus the limit shown.		Deviation Ratio of x		
.500	.500	1.000	.000	0.0000	0.0000	0.0000
.495	.505	.990	.010	0.0125	0.0157	0.0185
.490	.510	.980	.020	0.0250	0.0313	0.0371
.485	.515	.970	.030	0.0376	0.0384	0.0558
.480	.520	.960	.040	0.0502	0.0629	0.0744
.475	.525	.950	.050	0.0627	0.0786	0.0930
.470	.530	.940	.060	0.0753	0.0944	0.1116
.465	.535	.930	.070	0.0878	0.1100	0.1302
.460	.540	.920	.080	0.1004	0.1303	0.1489
.455	.545	.910	.090	0.1130	0.1416	0.1675
.450	.550	.900	.100	0.1256	0.1574	0.1862
.445	.555	.890	.110	0.1383	0.1733	0.2050
.440	.560	.880	.120	0.1510	0.1892	0.2239
.435	.565	.870	.130	0.1637	0.2052	0.2427
.430	.570	.860	.140	0.1764	0.2211	0.2615
.425	.575	.850	.150	0.1891	0.2370	0.2804
.420	.580	.840	.160	0.2018	0.2529	0.2992
.415	.585	.830	.170	0.2146	0.2690	0.3182
.410	.590	.820	.180	0.2375	0.2977	0.3521
.405	.595	.810	.190	0.2404	0.3013	0.3564
.400	.600	.800	.200	0.2533	0.3175	0.3755
.395	.605	.790	.210	0.2663	0.3338	0.3948
.390	.610	.780	.220	0.2793	0.3501	0.4141
.385	.615	.770	.230	0.2923	0.3663	0.4334
.380	.620	.760	.240	0.3054	0.3828	0.4528
.375	.625	.750	.250	0.3186	0.3993	0.4724
.370	.530	.740	.260	0.3318	0.4158	0.4919
.365	.635	.730	.270	0.3451	0.4325	0.5116
.360	.640	.720	.280	0.3584	0.4492	0.5314
.355	.645	.710	.290	0.3718	0.4660	0.5512
.350	.650	.700	.300	0.3853	0.4829	0.5712
.345	.655	.690	.310	0.3988	0.4998	0.5913
.340	.660	.680	.320	0.4125	0.5170	0.6116
.335	.665	.670	.330	0.4262	0.5342	0.6319
.330	.670	.660	.340	0.4399	0.5513	0.6522
.325	.675	.650	.350	0.4539	0.5689	0.6730
.320	.680	.640	.360	0.4678	0.5863	0.6936
.315	.685	.630	.370	0.4817	0.6037	0.7142
.310	.690	.620	.380	0.4958	0.6214	0.7351
.305	.695	.610	.390	0.5100	0.6392	0.7561
.300	.700	.600	.400	0.5244	0.6572	0.7775
.295	.705	.590	.410	0.5389	0.6754	0.7990
.290	.710	.580	.420	0.5534	0.6936	0.8205
.285	.715	.570	.430	0.5680	0.7119	0.8421
.280	.720	.560	.440	0.5829	0.7306	0.8642
.275	.725	.550	.450	0.5978	0.7492	0.8863
.270	.730	.540	.460	0.6128	0.7680	0.9085
.265	.735	.530	.470	0.6274	0.7863	0.9302
.260	.740	.520	.480	0.6433	0.8062	0.9538
.255	.745	.510	.490	0.6588	0.8257	0.9767
.250	.750	.500	.500	0.6745	0.8454	1.000
.245	.755	.490	.510	0.6903	0.8652	1.023
.240	.760	.480	.520	0.7063	0.8852	1.047
.235	.765	.470	.530	0.7225	0.9055	1.071
.230	.770	.460	.540	0.7389	0.9261	1.095
.225	.775	.450	.550	0.7554	0.9468	1.120
.220	.780	.440	.560	0.7722	0.9678	1.145
.215	.785	.430	.570	0.7893	0.9892	1.170
.210	.790	.420	.580	0.8066	1.011	1.196
.205	.795	.410	.590	0.8239	1.033	1.222
.200	.800	.400	.600	0.8416	1.055	1.248

Table 11-1 (concluded)
Limiting values of the various deviation ratios for given values of the normal probability functions

(NPDF)x for negative deviations	(NPDF)x for positive deviations	1 - (NMPF)x	(NMPF)x	$[(D(SD)R)x]_{(lim)}$	$[(D(AD)R)x]_{(lim)}$	$[(D(PD)R)x]_{(lim)}$
		Probability that (D)x will be				
		greater	less			
Probability that x will lie between minus infinity and		in magnitude than the limit shown, i.e., that x will lie		Limit expressed as Deviation-		
minus	plus	outside	inside	Standard	Average	Probable
the limit shown.		the range between minus and plus the limit shown.		Deviation Ratio of x		
.200	.800	.400	.600	0.8416	1.055	1.248
.195	.805	.390	.610	0.8597	1.078	1.275
.190	.810	.380	.620	0.8779	1.100	1.302
.185	.815	.370	.630	0.8965	1.124	1.329
.180	.820	.360	.640	0.9154	1.147	1.357
.175	.825	.350	.650	0.9346	1.171	1.386
.170	.830	.340	.660	0.9542	1.196	1.415
.165	.835	.330	.670	0.9742	1.221	1.444
.160	.840	.320	.680	0.9946	1.247	1.475
.155	.845	.310	.690	1.015	1.272	1.505
.150	.850	.300	.700	1.036	1.298	1.536
.145	.855	.290	.710	1.058	1.326	1.569
.140	.860	.280	.720	1.080	1.354	1.601
.135	.865	.270	.730	1.103	1.382	1.635
.130	.870	.260	.740	1.126	1.411	1.669
.125	.875	.250	.750	1.150	1.441	1.705
.120	.880	.240	.760	1.175	1.473	1.742
.115	.885	.230	.770	1.201	1.505	1.781
.110	.890	.220	.780	1.226	1.537	1.818
.105	.895	.210	.790	1.253	1.570	1.858
.100	.900	.200	.800	1.282	1.607	1.901
.095	.905	.190	.810	1.312	1.644	1.945
.090	.910	.180	.820	1.341	1.681	1.988
.085	.915	.170	.830	1.372	1.720	2.034
.080	.920	.160	.840	1.405	1.761	2.083
.075	.925	.150	.850	1.440	1.805	2.135
.070	.930	.140	.860	1.476	1.850	2.188
.065	.935	.130	.870	1.514	1.897	2.245
.060	.940	.120	.880	1.555	1.949	2.305
.055	.945	.110	.890	1.598	2.003	2.363
.050	.950	.100	.900	1.645	2.062	2.439
.045	.955	.090	.910	1.696	2.126	2.515
.040	.960	.080	.920	1.751	2.195	2.596
.035	.965	.070	.930	1.811	2.270	2.685
.030	.970	.060	.940	1.881	2.357	2.789
.025	.975	.050	.950	1.960	2.456	2.906
.0225	.9775	.045	.955	2.005	2.513	2.973
.020	.980	.040	.960	2.054	2.574	3.045
.0175	.9825	.035	.965	2.108	2.642	3.125
.015	.985	.030	.970	2.170	2.720	3.217
.0125	.9875	.025	.975	2.240	2.807	3.321
.010	.990	.020	.980	2.327	2.916	3.450
.0075	.9925	.015	.985	2.430	3.046	3.603
.005	.995	.010	.990	2.576	3.229	3.819
.0045	.9955	.009	.991	2.612	3.274	3.873
.004	.996	.008	.992	2.652	3.324	3.932
.0035	.9965	.007	.993	2.697	3.380	3.988
.003	.997	.006	.994	2.748	3.444	4.074
.0025	.9975	.005	.995	2.800	3.509	4.151
.002	.998	.004	.996	2.878	3.607	4.267
.0015	.9985	.003	.997	2.967	3.719	4.399
.001	.999	.002	.998	3.090	3.873	4.581
.0005	.9995	.001	.999	3.290	4.123	4.878
.000,4	.999,6	.0008	.9992	3.35	4.20	4.96
.000,25	.999,75	.0005	.9995	3.48	4.36	5.15
.000,1	.999,9	.0002	.9998	3.71	4.65	5.50
.000,05	.999,95	.0001	.9999	3.89	4.88	5.76
.000,04	.999,96	.000,08	.999,92	3.96	4.96	5.86
.000,025	.999,975	.000,05	.999,95	4.08	5.11	6.04
.000,01	.999,99	.000,02	.999,98	4.28	5.35	6.34
.000,005	.999,995	.000,01	.999,99	4.43	5.55	6.56
		10^{-6}	$1 - 10^{-6}$	4.90	6.14	7.26
		10^{-7}	$1 - 10^{-7}$	5.33	6.68	7.90
		10^{-8}	$1 - 10^{-8}$	5.74	7.19	8.50

Table 11-2
Normal probability functions for given limiting values of the deviation-standard deviation ratio

[(D(SD)R)x](lim)	(NMPF)x	1 - (NMPF)x	(NPDF)x	(NPDF)x
Limit expressed as Deviation-Standard Deviation Ratio of x	Probability that (D)x will be		for positive deviations	for negative deviations
	less	greater		
	in magnitude than the limit shown, i.e., that x will lie		Probability that x will lie between minus infinity and	
	inside	outside	plus	minus
	the range between minus and plus the limit shown.		the limit shown.	
0.0	.0000	1.0000	.5000	.5000
0.1	.0797	.9203	.5398	.4602
0.2	.1585	.8415	.5793	.4207
0.3	.2358	.7642	.6179	.3821
0.4	.3108	.6892	.6554	.3446
0.5	.3829	..6171	.6915	.3085
0.6	.4515	.5485	.7258	.2742
0.7	.5161	.4839	.7580	.2420
0.8	.5763	.4237	.7881	.2119
0.9	.6319	.3681	.8159	.1841
1.0	.6827	.3173	.8413	.1587
1.1	.7287	.2713	.8643	.1357
1.2	.7699	..2301	.8849	.1151
1.3	.8064	.1936	.9032	.0968
1.4	.8385	.1615	.9192	.0808
1.5	.8664	.1336	.9332	.0668
1.6	.8904	.1096	.9452	.0548
1.7	.9109	.0891	.9554	.0446
1.8	.9281	..0717	.9641	.0359
1.9	.9426	.0574	.9713	.0287
2.0	.9545	.0455	.9773	.0227
2.1	.9643	.0357	.9821	.0179
2.2	.9722	.0278	.9861	.0139
2.3	.9786	.0214	.9893	.0107
2.4	.9836	.0164	.9918	.00820
2.5	.9876	.0124	.9938	.00621
2.6	.9907	.0093	.99534	.00466
2.7	.99307	.00693	.99653	.00347
2.8	.99489	.00511	.99744	.00256
2.9	.99627	.00373	.99813	.00187
3.0	.99730	.00270	.99865	.00135
3.5	.99953	.00047	.99977	.00023
4.0		6.7×10^{-5}		3.4×10^{-5}
4.5		7.1×10^{-6}		3.6×10^{-6}
5.0		6.0×10^{-7}		3.0×10^{-7}

Table 11-3
Normal probability functions for given limiting values of the deviation-average deviation ratio

$[(D/AD)R)x]_{(lim)}$	(NMPF)x	1 - (NMPF)x	(NPDF)x	(NPDF)x
Limit expressed as Deviation-Average Deviation Ratio of x	Probability that (D)x will be		for positive deviations	for negative deviations
	less	greater		
	in magnitude than the limit shown, i.e., that x will lie		Probability that x will lie between minus infinity and	
	inside	outside	plus	minus
	the range between minus and plus the limit shown.		the limit shown.	
0.0	.0000	1.0000	.5000	.5000
0.1	.0636	.9364	.5318	.4682
0.2	.1268	.8732	.5634	.4366
0.3	.1892	.8108	.5946	.4054
0.4	.2504	.7496	.6252	.3748
0.5	.3101	.6899	.6550	.3450
0.6	.3679	.6321	.6839	.3161
0.7	.4235	.5765	.7117	.2993
0.8	.4767	.5233	.7384	.2616
0.9	.5273	.4727	.7636	.2364
1.0	.5751	.4249	.7875	.2125
1.1	.6199	.3801	.8099	.1901
1.2	.6617	.3383	.8308	.1692
1.3	.7004	.2996	.8502	.1498
1.4	.7360	.2640	.8680	.1320
1.5	.7686	.2314	.8843	.1157
1.6	.7983	.2017	.8991	.1009
1.7	.8250	.1750	.9125	.0875
1.8	.8490	.1510	.9245	.0755
1.9	.8705	.1295	.9352	.0648
2.0	.8895	.1105	.9447	.0553
2.1	.9062	.0938	.9531	.0469
2.2	.9208	.0792	.9604	.0396
2.3	.9335	.0665	.9668	.0332
2.4	.9445	.0555	.9723	.0277
2.5	.9539	.0461	.9770	.0230
2.6	.9620	.0380	.9810	.0190
2.7	.9688	.0312	.9844	.0156
2.8	.9745	.0255	.9873	.0127
2.9	.9793	.0207	.9897	.0103
3.0	.9833	.0167	.9917	.0083
3.1	.9866	.0134	.9933	.0067
3.2	.9893	.0107	.9947	.0053
3.3	.9915	.0085	.9958	.0042
3.4	.99333	.00667	.99667	.00333
3.5	.99477	.00523	.99739	.00261
3.6	.99593	.00407	.99797	.00203
3.7	.99684	.00316	.99842	.00158
3.8	.99757	.00243	.99879	.00121
3.9	.99814	.00186	.99907	.00093
4.0	.99859	.00141	.99930	.00070
4.5		3.3×10^{-4}		1.7×10^{-4}
5.0		7.1×10^{-5}		3.6×10^{-5}
5.5		1.2×10^{-5}		6.0×10^{-6}
6.0		1.7×10^{-6}		9.0×10^{-7}
6.5		2.1×10^{-7}		1.1×10^{-7}

Table 11-4

Normal probability functions for given limiting values of the deviation-probable deviation ratio

[(D/PD)R)x]$_{\text{lim}}$)	(NMPF)x	1 - (NMPF)x	(NPDF)x	(NPDF)x
Limit expressed as Deviation-Probable Deviation Ratio of x	Probability that (D)x will be less	greater	for positive deviations	for negative deviations
	in magnitude than the limit shown, i.e., that x will lie		Probability that x will lie between minus infinity and	
	inside	outside	plus	minus
	the range between minus and plus the limit shown.		the limit shown.	
0.0	.0000	1.0000	.5000	.5000
0.1	.0539	.9461	.5269	.4731
0.2	.1073	.8927	.5536	.4464
0.3	.1604	.8396	.5802	.4198
0.4	.2127	.7873	.6063	.3937
0.5	.2641	.7359	.6320	.3680
0.6	.3143	.6857	.6571	.3429
0.7	.3632	.6368	.6816	.3184
0.8	.4105	.5895	.7052	.2948
0.9	.4561	.5439	.7280	.2720
1.0	.5000	.5000	.7500	.2500
1.1	.5419	.4581	.7709	.2291
1.2	.5817	.4183	.7908	.2092
1.3	.6194	.3806	.8097	.1903
1.4	.6550	.3450	.8275	.1725
1.5	.6883	.3117	.8441	.1559
1.6	.7195	.2805	.8597	.1403
1.7	.7485	.2515	.8742	.1258
1.8	.7753	.2247	.8876	.1124
1.9	.8000	.2000	.9000	.1000
2.0	.8227	.1773	.9113	.0887
2.1	.8434	.1566	.9217	.0783
2.2	.8622	.1378	.9311	.0689
2.3	.8792	.1208	.9396	.0604
2.4	.8945	.1055	.9472	.0528
2.5	.9083	.0917	.9541	.0459
2.6	.9205	.0795	.9602	.0398
2.7	.9314	.0686	.9657	.0343
2.8	.9411	.0589	.9705	.0295
2.9	.9496	.0504	.9748	.0252
3.0	.9570	.0430	.9785	.0215
3.1	.9635	.0365	.9817	.0183
3.2	.9691	.0309	.9845	.0155
3.3	.9740	.0260	.9870	.0130
3.4	.9782	.0218	.9891	.0109
3.5	.9818	.0182	.9909	.0091
3.6	.9848	.0152	.9924	.0076
3.7	.9874	.0126	.9937	.0063
3.8	.9896	.0104	.9948	.0052
3.9	.9915	.0085	.9957	.0043
4.0	.99302	.00698	.99651	.00349
4.1	.99432	.00568	.99716	.00284
4.2	.99538	.00462	.99769	.00231
4.3	.99627	.00373	.99813	.00187
4.4	.99700	.00300	.99850	.00150
4.5	.99760	.00240	.99880	.00120
5.0		7.6×10^{-4}		3.8×10^{-4}
5.5		2.1×10^{-4}		1.1×10^{-4}
6.0		5.5×10^{-5}		2.3×10^{-5}
6.5		1.2×10^{-5}		6.0×10^{-6}
7.0		2.5×10^{-6}		1.3×10^{-6}

GROUPED DATA

11.32. Grouped data is defined as a set of typical value items arranged in steps of equal increments of value, so that several items may fall within a single step. Grouped data is used in two different situations. In one, the smallest observed increment is relatively large in comparison with the range of values, so that several items necessarily fall at the same observed value. For example, a noise voltage is read to the nearest millivolt, and thirty readings fall in a range of twenty millivolts. In the other, there is a very large number of items, and computing work can be simplified by grouping them in somewhat larger increments of value. For example, a thousand freshmen are weighed to the nearest half pound, but the weights are grouped in five-pound steps to simplify computing the average and the standard deviation.

11.33. Define the value of the k^{th} step as x_k, and the increment as Δx, i.e., $x_{(k+1)} = x_k + \Delta x$. Define the boundaries of the k^{th} step as $(x_k - \Delta x/2)$ and $(x_k + \Delta x/2)$, i.e., half way between the defined values of successive steps. Note that in some cases the boundaries and the step values work out to odd amounts. For example, if the freshmen were weighed to the nearest half pound and grouped in steps of five pounds, the steps might be given as 120-1/2 to 125 lb, 125-1/2 to 130 lb, etc., but the boundaries would really be 120-1/4 lb (half way between 120 lb and 120-1/2), 125-1/4 lb, 130-1/4 lb, etc., and the values of the steps (half way between these boundaries) would be 122-3/4 lb, 127-3/4 lb, etc.

11.34. It is more convenient to use the value of a step times the number of items falling in it, in computing an average, than to use all separate values of the items; i.e., instead of computing

$$(Av)x = \frac{1}{n}\sum_{i=1}^{n} x_i \quad \text{and} \quad (SD)x = \sqrt{\frac{1}{n}\sum_{i=1}^{n}[(D)x_i]^2}. \qquad \begin{matrix}(11\text{-}1),\\ (11\text{-}4)\end{matrix}$$

It is easier to compute

$$(Av)x \approx \frac{1}{n}\sum_{k=1}^{m} n_k x_k \quad \text{and} \quad (SD)x \approx \sqrt{\frac{1}{n}\sum_{k=1}^{m}[n_k (D)x_k]^2} \qquad \begin{matrix}(11\text{-}32),\\ (11\text{-}33)\end{matrix}$$

where n is the total number of items, m is the number of steps, and n_k is the number of items in the k^{th} step. The error introduced by this approximation is negligible when n and m are reasonably large.

11.35. Grouped data may be plotted as a frequency diagram (also called a histogram), n_k against x_k. But as will be seen in Fig. 11-7 parts a) and b), such a diagram yields little or no information unless the total number of items is quite large. On the other hand, if grouped data is plotted as a distribution diagram, a

ITEM DATA

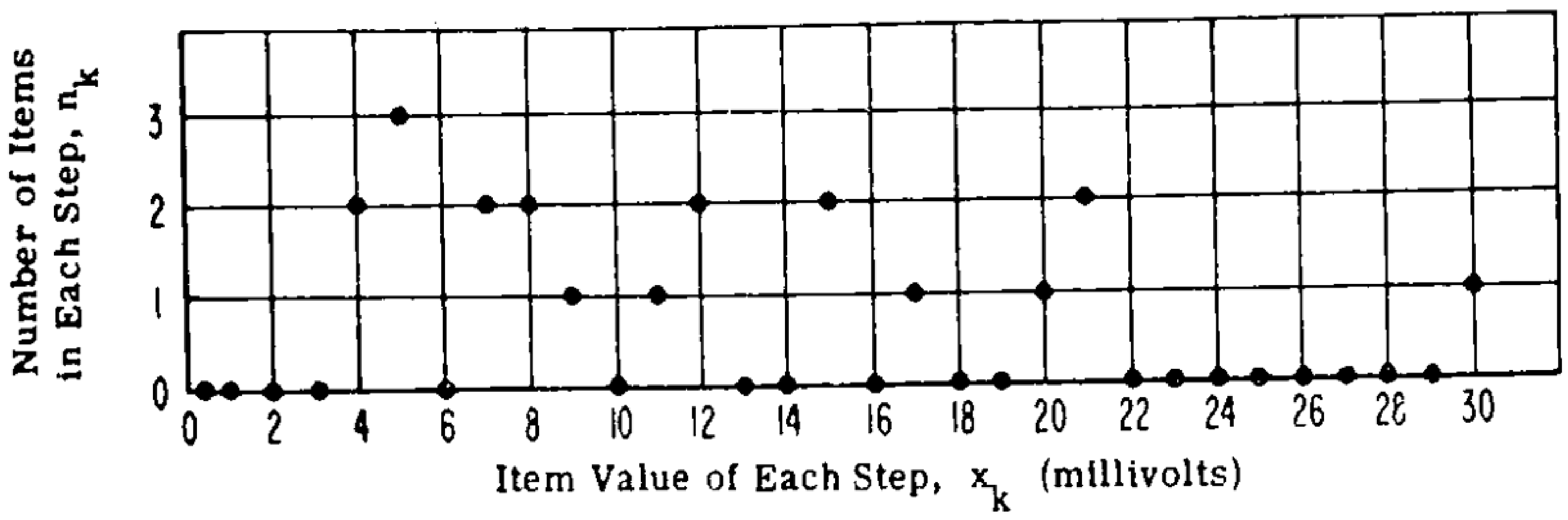
(Value of Residual Voltage at the Null Point for 20 samples of an a.c. Balancing Device)

Number	Value (millivolts)	Number	Value (millivolts)
1	15	11	12
2	21	12	8
3	4	13	15
4	7	14	20
5	5	15	11
6	21	16	5
7	5	17	30
8	4	18	7
9	12	19	8
10	17	20	9

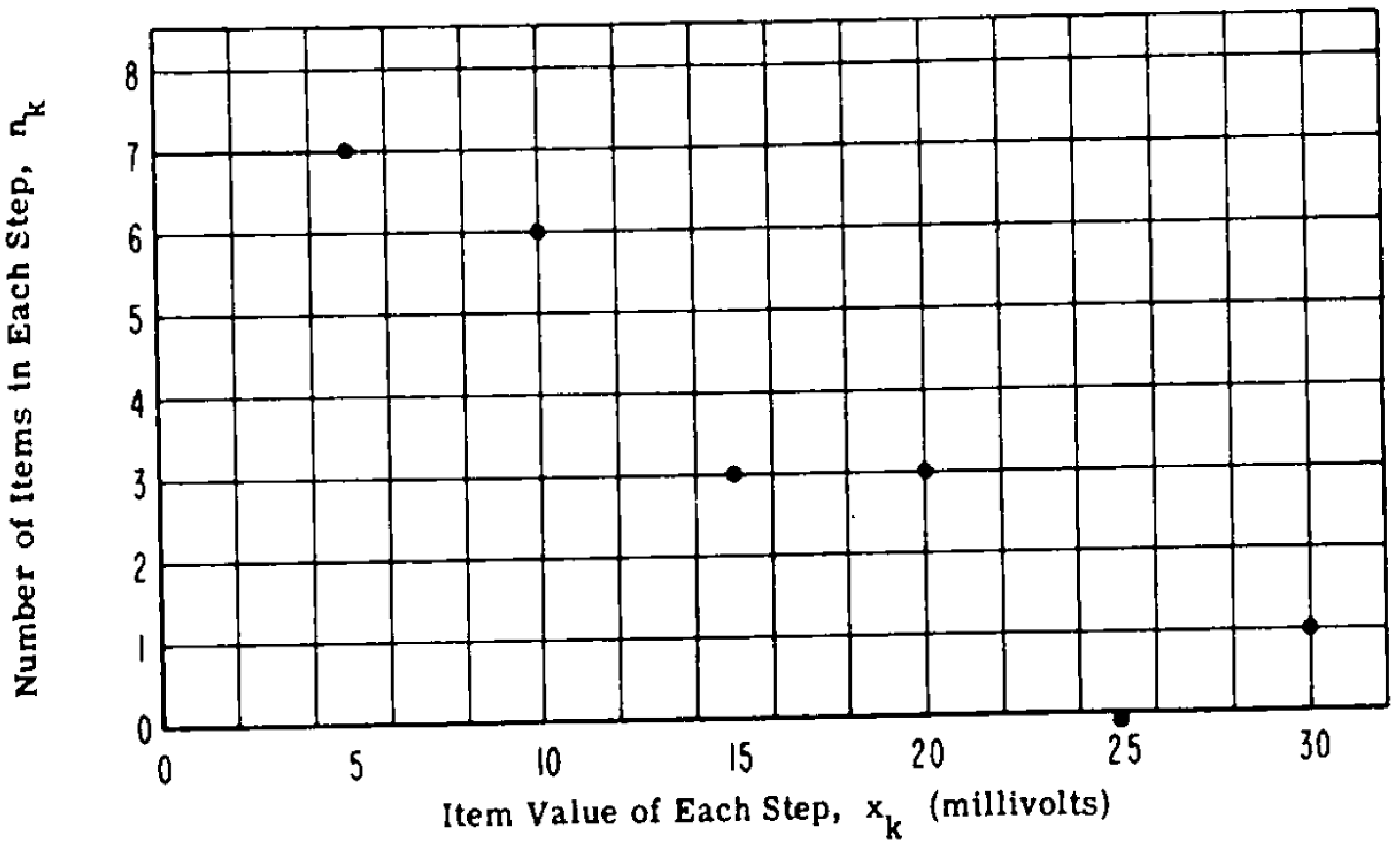
GROUPED DATA

(Calculations for Frequency and Distribution Diagrams from Item Data)

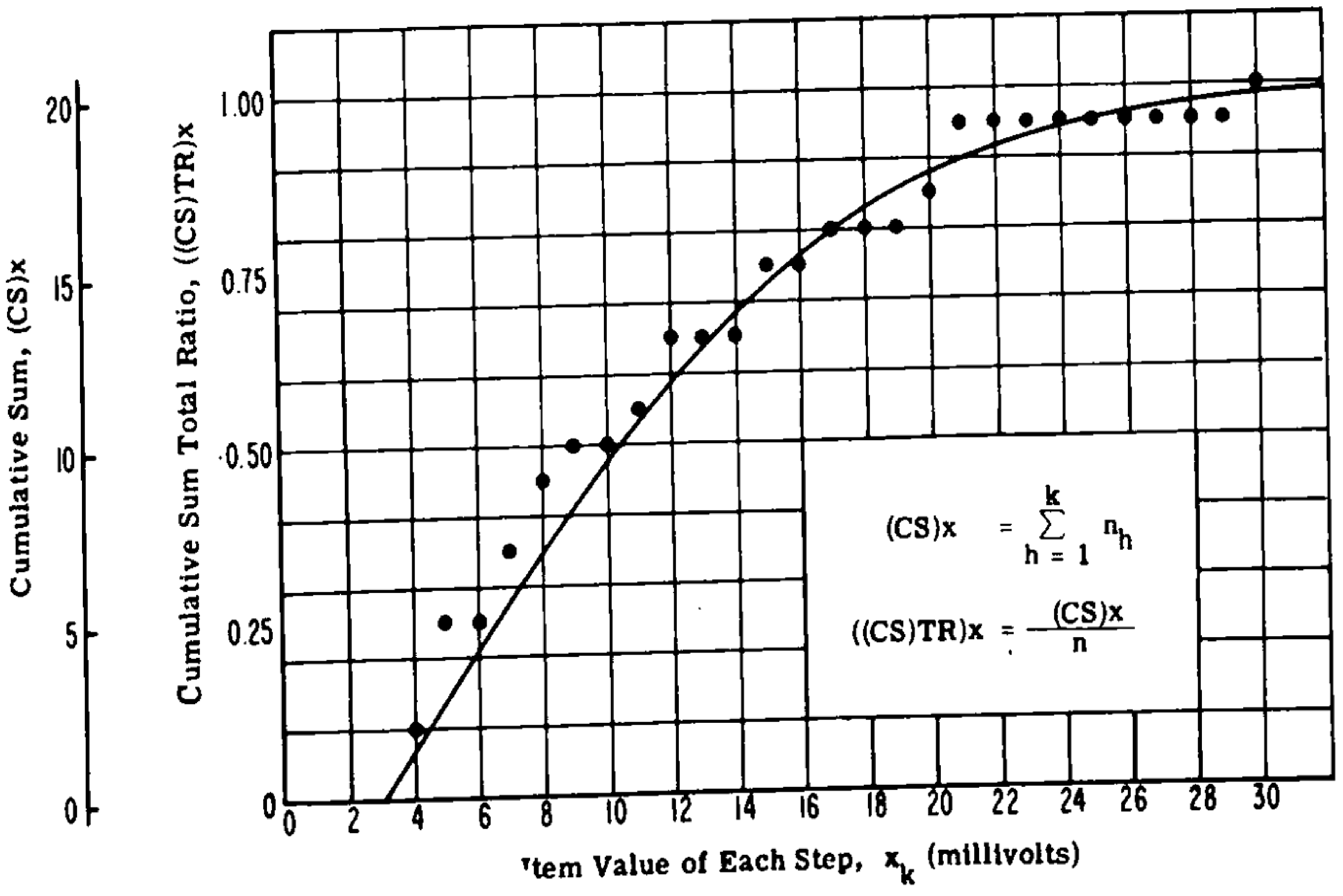
Item Value, x_k	Number of Items In Step Interval, n_k (Step interval is one millivolt)	Number of Items In Step Interval, n_k (Step interval is five millivolts)	Cumulative Sum, $(CS)n_k$ (Step interval is one millivolt)	Cumulative Sum Total Ratio, $(CSTR)n_k$ (Step interval is one millivolt)
1	0		0	0
2	0		0	0
3	0	7	0	0
4	2		2	.10
5	3		5	.25
6	0		5	.25
7	2		7	.35
8	2	6	9	.45
9	1		10	.50
10	0		10	.50
11	1		11	.55
12	2		13	.65
13	0	3	13	.65
14	0		13	.65
15	2		15	.75
16	0		15	.75
17	1		16	.80
18	0	3	16	.80
19	0		16	.80
20	1		17	.85
21	2		19	.95
22	0		19	.95
23	0	0	19	.95
24	0		19	.95
25	0		19	.95
26	0		19	.95
27	0		19	.95
28	0	1	19	.95
29	0		19	.95
30	1		20	1.00
31				
32				



a) Frequency diagram for steps of one millivolt



b) Frequency diagram for steps of five millivolts



c) Distribution diagram for steps of one millivolt

Fig. 11-7. Example of grouped data plots.

small number of items will yield adequate information. Define (for each step) the cumulative sum of x as the sum of the number of items in all steps from the farthest negative to the k^{th} step inclusive, i.e.,

$$(CS)x \equiv \sum_{h=1}^{k} n_h. \tag{11-34}$$

Define (for each step) the cumulative sum — total ratio of x as the ratio of the cumulative sum of x to the total number of items, i.e.,

$$((CS)TR)x \equiv \frac{(CS)x}{n}, \tag{11-35}$$

where n is the total number of items. A plot of either (CS)x or ((CS)TR)x against x_k forms a distribution diagram. ((CS)TR)x has the advantage of being a dimensionless ratio running from zero to unity; hence it can be compared directly with the normal probability distribution function or the normal magnitude probability function, whichever is appropriate. It is often convenient, as shown in Fig. 11-7c, to plot (CS)x directly and to add a scale of ((CS)TR)x on the diagram.

Example: Twenty a-c balancing devices are taken from the production line. Each is adjusted to its null point and the residual output voltage is measured to the nearest millivolt, with the results as tabulated in Fig. 11-7. The problem is to fit a normal curve to the data. This residual voltage is known to consist of out-of-phase, harmonic, and noise components, which add on an a-c voltmeter so that the reading is the sum of their magnitudes without regard to phase or sign. Hence the normal magnitude probability function can be expected to fit the data.

11.36. Figure 11-7a is a frequency diagram for $\Delta x = 1$ millivolt. It is so rough that it gives no useful information. Figure 11-7b is a similar diagram for $\Delta x = 5$ millivolts. Here the frequency trend is observable, but it would be difficult to fit a smooth curve to the points. Figure 11-7c is a distribution diagram for $\Delta x = 1$ millivolt. Vertical scales for both (CS)x and ((CS)TR)x are given on the same diagram. Here it is evident that the smooth curve shown fits well. This curve is (NMPF)x for (Av)x = 3 millivolts, and (AD)x = 8.8 millivolts. The data is interpreted as meaning that there is a constant noise voltage of 3 millivolts in all units plus other components varying from 1 to 27 millivolts in different units and following the normal probability law. The value of (AD)x was obtained by taking the average of the 20 voltage readings, which is 11.8 millivolts, and subtracting the apparent noise level of 3 millivolts.

TREATMENT OF LARGE DEVIATIONS

11.37. When a set of data contains one or more items whose deviations appear to be excessively large, one of two types of treatment may be called for. If the data is such that some of the variations may be significant, i.e., due to causes other than uncertainties, the treatment is designed to separate significant deviations from those due to uncertainties alone. If the data is not expected to contain significant variations, the treatment is designed to weed out any deviations so large and so unlikely that they make the average of the set less accurate.

11.38. The test for a significant deviation is to select a confidence figure, and then arbitrarily assume that any deviation outside the corresponding limits is significant. The usual choice is CF = 0.95, making the limits 2(SD)x, or 2-1/2 (AD)x.

Example: A machine is producing steel balls. A sample of 100 balls is taken each day, and the diameter of each ball of the hundred is measured. The average diameter over a long period is 0.25000 inch. The average deviation of individual balls over this period is 0.00040 inch. If deviations beyond the 95 percent confidence limits are considered significant of a change in the adjustment of the machine, which of the following daily averages are thus significant?

Monday	0.25008
Tuesday	0.25012
Wednesday	0.24996
Thursday	0.24989
Friday	0.25001
Saturday	0.24999

Since

$$(AD)x = 0.00040$$

$$(AD)\,[(Av)x]_{100} = \frac{0.00040}{\sqrt{100}} = 0.00004$$

$$\text{95 percent point is } 2.5\,(AD)[(Av)x]_{100} = 0.00010.$$

The days when $|(D)\,[(Av)x]_{100}| > 0.00010$ are Tuesday and Thursday; i.e., only Tuesday's and Thursday's averages signify a change from the normal adjustment.

11.39. The test for rejecting large deviations when they decrease the accuracy of the average of a set of data is called Cauchy's criterion. It is:

> *If a set of n values contains one with a deviation so large that its probability of occurrence is less than 1/2 n, this value is to be discarded.*

11.40. To apply Cauchy's criterion to a set of n values, compute (Av)x for the set, (D)x for each value, and (SD)x or (AD)x for the set. From the tables find the (NMPF)x for the largest magnitude of (D(SD)R)x or (D(AD)R)x, and if this shows 1 – (NMPF)x < 1/2n, discard this one value and use the remaining (n - 1) values as a new set. Apply the same criterion to the largest magnitude deviation in the new set, and if it is rejected, use the remaining (n - 2) values. Continue until a set is found in which the largest deviation is not rejected.

11.41. Figure 11-8 presents Cauchy's criterion in graphical form. The non-linear scale of n is computed as [1 – (NMPF)x] = 1/2n, referred to linear scales of (D(SD)R)x and (D(AD)R)x. Thus any magnitude of (D(SD)R)x or (D(AD)R)x which has a probability of occurrence less than 1/2n will lie above the scale position of n in this figure.

11.42. Certain short cuts may be used in computing. When computing (SD)x with a slide-disk calculator, it is not necessary to compute all deviations, but only those which are suspiciously large. If one value, x_k, is discarded, it is not necessary to go through the entire work of computing $[(Av)x]_{(n-1)}$, $[(AD)x]_{(n-1)}$, or $[(SD)x]_{(n-1)}$ for the new set of (n-1) values. These can be obtained from the computations for the old set of n values by the following formulas

$$[(AD)x]_{(n-1)} = \frac{1}{n-1} \sum_{i=1}^{(n-1)} \left| [(D)x_i]_n + \frac{[(D)x_k]_n}{(n-1)} \right| \qquad (11\text{-}36)$$

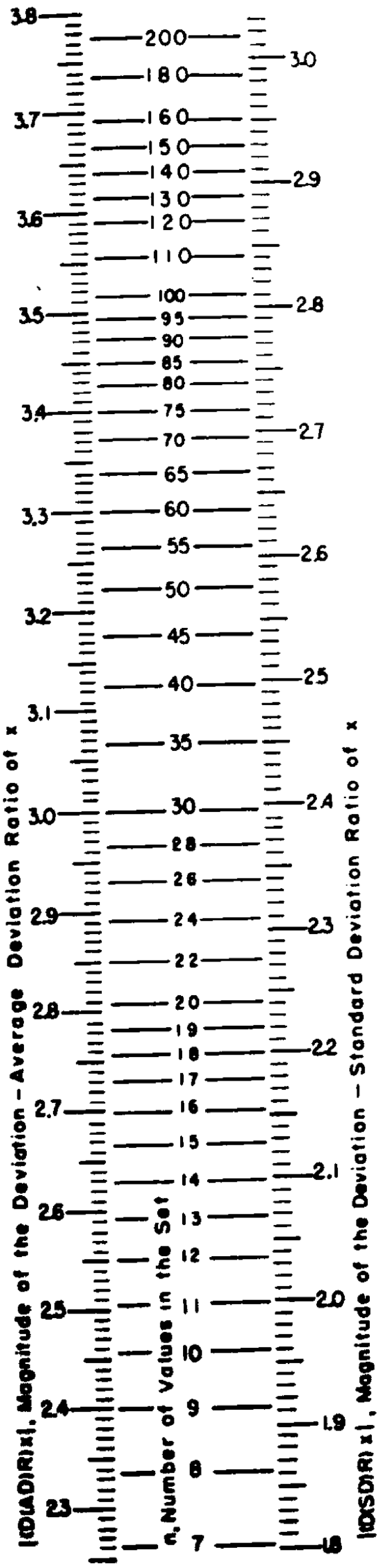
This can be computed conveniently by summing separately deviations from $[(Av)x]_n$ which will stay positive, deviations which will stay negative, and deviations which will change in sign because they are opposite in sign to, and smaller in magnitude than, $[(D)x_k]_n/n-1$. Then add algebraically to each sum $[(D)x_k]_n/n-1$ multiplied by the number of terms in that sum; add the magnitudes of the three results, and divide by (n-1).

$$[(Av)x]_{(n-1)} = \frac{1}{n-1} \left[n[(Av)x]_n - x_k \right] \qquad (11\text{-}37a)$$

$$= \frac{1}{n-1} \left[\left(\sum x_i \right) - x_k \right] \qquad (11\text{-}37b)$$

$$[(SD)x]^2_{(n-1)} = \frac{1}{n-1} \left[n[(SD)x]^2_n - \frac{1}{n-1}[(D)x_k]^2 \right] \qquad (11\text{-}38)$$

11.43. To use this last formula with a slide-disk calculator, take the answer on the vertical scale before applying the nomogram of the calculator, or take the standard deviation and work backward through the nomogram to locate this point, which is $[(SD)x]\sqrt{n}$. Place the disk indicator mark at this point. Locate $(D)x_k$ on the standard deviation scale of the nomogram, pass a line through n and momentarily locate a point on the vertical scale. Then pivot the line at this point, so it passes through (n-1); where it crosses the standard deviation scale is then



METHOD OF USING THESE SCALES:

If the largest magnitude of (D(SD)R)x or (D(AD)R)x lies above the line for the number of values, n, in the set, discard the corresponding value and use the remaining (n-1) values as a new set.

Fig. 11-8. Scale for applying Cauchy's criterion.

$(D)x_k \sqrt{n/(n-1)}$. Place zero on the horizontal scale at the index. Insert the stylus in the disk at the index and move it to $(D)x_k \sqrt{n/(n-1)}$ on the horizontal scale. This performs the operation in the large brackets. Use the nomogram from the new indicator position through (n-1), obtaining the desired answer, $[(SD)x]_{(n-1)}$.

11.44. In comparing the significance test and Cauchy's criterion, it should be noted that if the significance test is applied to single values, it uses an arbitrary confidence figure which is independent of the number of values in the set, where Cauchy's criterion makes the confidence figure a function of the number of values in the set, i.e.,

$$(CF) = 1 - \frac{1}{2n} \qquad (11\text{-}39)$$

If the significance test is applied to the average of a set of n values, the standard deviation of this average decreases as n increases, making the region of non-significance correspondingly smaller.

Example: If the set of twelve vacuum tubes described in previous examples included a thirteenth tube with a plate current of 42.7 ma, should this value be rejected under Cauchy's criterion?

For the set of thirteen plate currents,

$$(Av)x = 39.24 \text{ ma}$$

$$(SD)x = 1.41 \text{ ma}$$

For this thirteenth tube,

$$(D)x = 42.7 - 39.24 = +3.46 \text{ ma}$$

$$(D(SD)R)x = 3.46/1.41 = 2.37$$

Since this is far above the scale point for n = 13 in Fig. 11-8, this value should be rejected.

11.45. Correlation deals with any pair of variables which may have a partial functional interrelationship. For example, let

$$x = \text{a variable}$$

and

$$y = \text{a variable which may be partially dependent on } x.$$

Assume that n concurrent pairs of typical values of x and y are known, i.e.,

$$(x_i, y_i) \Big|_{i=1}^{i=n}$$

11.46. Two variables which illustrate this sort of relationship are the heights and weights of a group of individuals. It is well known that while the average weight increases with height, there are wide individual variations in weight for a given height. Table 11-5 gives the heights and weights of fifty male students selected at random from the files of the Medical Department, Massachusetts Institute of Technology.

11.47. The regression line of y on x is defined as the linear function of x which best fits y by the "least square" criterion, i.e., it is a straight line in the x, y plane so chosen that the sum of the squares of the y_i deviations from it is a minimum. In equation form

$$y_{(rgr)} = K_o + S_{(rgr)[x;y]}(D)x \tag{11-40}$$

where

$y_{(rgr)}$ = value of y on the regression line

$(D)x$ = $x - (Av)x$

$S_{(rgr)[x;y]}$ = slope of the regression line

K_o = constant which locates the line.

11.48. A regression line is useful where y is believed or is assumed to be the sum of two components, one linearly dependent on x and the other independent of x. In this case the regression line, $y_{(rgr)}$, is our best estimate of the component dependent on x, and the deviation from this line, $[(D)y_i]_{(rgr)}$, is our best estimate of the component independent of x.

11.49. The values K_o and $S_{(rgr)[x;y]}$ of the regression line equation are to be chosen to minimize the sum of the squared deviations, $\sum_{i=1}^{n} [(D)y_i]^2_{(rgr)}$ where

$$[(D)y_i]_{(rgr)} \equiv y_i - y_{(rgr)(x_i)} .$$

The condition for a minimum value of this sum is that

$$\frac{\partial \Sigma[(D)y_i]^2_{(rgr)}}{\partial S_{(rgr)[x;y]}} = 0 \qquad (11\text{-}41)$$

and

$$\frac{\partial \Sigma[(D)y_i]^2_{(rgr)}}{\partial K_0} = 0 \qquad (11\text{-}42)$$

It can be shown that this condition is satisfied when

$$S_{(rgr)[x;y]} = \frac{(Av)[y_i(D)x_i]}{(Av)[(D)x_i]^2} \qquad (11\text{-}43)$$

and

$$K_0 = (Av)y_i \qquad (11\text{-}44)$$

It can further be shown that the following forms are equivalent and may be used interchangeably wherever computing will be simplified

$$\begin{aligned}(Av)[y_i(D)x_i] &= (Av)[x_i(D)y_i] & (11\text{-}45a)\\ &= (Av)[(D)x_i(D)y_i] & (11\text{-}45b)\\ &= (Av)(x_i, y_i) - (Av)x_i(Av)y_i & (11\text{-}45c)\\ &= (Av)[(x_i - K_1)(y_i - K_2)] & \\ &\quad - [K_1 - (Av)x_i][K_2 - (Av)y_i] & (11\text{-}45d)\end{aligned}$$

where K_1 and K_2 are any convenient constants.

$$\begin{aligned}(Av)[(D)x_i]^2 &= [(SD)x_i]^2 & (11\text{-}46a)\\ &= (Av)(x_i)^2 - [(Av)x_i]^2 & (11\text{-}46b)\\ &= (Av)(x_i - K_3)^2 - [K_3 - (Av)x_i]^2 & (11\text{-}46c)\end{aligned}$$

where K_3 is any convenient constant.

11.50. As a practical matter, the forms in (11-45c) and (11-46b) represent the difference between two nearly equal quantities, and hence each quantity must be carried to a moderately large number of significant figures, which requires a calculating machine. The forms in (11-45d) and (11-46c) are convenient for manual computation. If the constants are chosen as whole numbers close to the averages of the variables, the differences $(x_i - K_1)$, etc., are small whole numbers or simple fractions which are easy to multiply, and the effect of the usual odd decimal values of the averages is confined to the small second term. As an example, the computation using (11-45d) is given in the last four columns of Table 11-5.

TABLE VII-I

Correlation of Heights and Weights of Fifty M.I.T. Students

Columns 2 and 3 give the initial data as furnished by the M.I.T. Medical Department. Columns 4, 5, 6, and 7 show the steps in obtaining the correlation numerator, (Av)[(D)h(D)w], by Eq. (11-45d).

(1) No.	(2) h Height In.	(3) w Weight Lb	(4) (h-69)	(5) (w-155)	(6) Positive Products	(7) Negative Products
1	68	140	-1	-15	+ 15	
2	70 3/4	153	+1 3/4	- 2		- 3 1/2
3	69 1/2	153	+ 1/2	- 2		- 1
4	71 1/2	138	+1 1/2	-17		-25 1/2
5	68	145	-1	-10	+ 10	
6	68 1/2	151	- 1/2	- 4	+ 2	
7	68	170 1/2	-1	+15 1/2		-15 1/2
8	70	181	+1	+26	+ 26	
9	72	154 1/2	+3	- 1/2		- 1 1/2
10	72	138	+3	-17		-51
11	72 1/2	134	+3 1/2	-21		-72 1/2
12	64	142	-5	-13	+ 65	
13	69	150	0	- 5	0	
14	70	160	1	+ 5	+ 5	
15	65 1/2	150	3 1/2	- 5	+ 17 1/2	
16	66 3/4	144	2 1/2	-11	+ 27 1/2	
17	66	150	3	- 5	+ 15	
18	68 3/4	128	1/4	-27	+ 6 3/4	
19	69 1/2	125	1/2	-30		-15
20	67	145	2	-10	+ 20	
21	67 1/2	133 1/2	1 1/2	-21 1/2	+ 32 1/4	
22	69 1/4	195	1/4	+40	+ 10	
23	71	150	2	- 5		-10
24	74	159	+5	+ 4	+ 20	
25	67	165	-2	+10		-20
26	70 1/4	131	+1 1/4	-25		-31 1/4
27	70 1/2	154 1/2	+1 1/2	- 1/2		- 3/4
28	73	148	+4	- 7		-28
29	72	175 1/2	+3	+20 1/2	+ 61 1/2	
30	69	147	0	- 8	0	
31	70 1/2	188	+1 1/2	+33	+ 49 1/2	
32	70	172 1/2	+1	+17 1/2	+ 17 1/2	
33	67 1/2	144 1/2	-1 1/2	-10 1/2	+ 15 3/4	
34	71	155	+2	0	0	
35	74	157	+5	+ 2	+ 10	
36	71 1/2	180	+2 1/2	+25	+ 62 1/2	
37	75	205	+6	+50	+300	
38	73	143	+4	-12		-48
39	66	163 1/2	-3	+ 8 1/2		-25 1/2
40	66 1/2	132	-2 1/2	-23	+ 57 1/2	
41	67	141	-2	-14	+ 28	
42	70	192	+1	+37	+ 37	
43	70	157	+1	+ 2	+ 2	
44	71	172	+2	+17	+ 34	
45	69	153	0	- 2	0	
46	72 1/2	168	+3 1/2	+13	+ 55 1/2	
47	71	176	+2	+21	+ 42	
48	60	142	-9	-13	+117	
49	69	152	0	- 3	0	
50	70	154	+1	- 1		- 1
	3475 1/4	**7758**			**+1161 3/4**	**-350**

−350

50)811.75

16.2350

(Av)h = 69.505 in.

(SD)h = 4.262 in.

$[(SD)h]^2$ = 18.166

(Av)w = 155.16 lb

(SD)w = 24.80 lb

$[(SD)w]^2$ = 614.99

$$69 - (Av)h = -0.505$$
$$155 - (Av)w = -0.160$$
$$-0.505 \times -0.160 = 0.0808$$
$$16.2350 - 0.0808 = 16.1542 = (Av)[(D)h(D)w]$$

$$S_{(rgr)(w,h)} = \frac{16.1542}{614.99} = 0.0263 \text{ in/lb}$$

$$S_{(rgr)(h,w)} = \frac{16.1542}{18.166} = 0.889 \text{ lb/in}$$

$$(CC)[w,h] = \frac{16.1542}{4.262 \times 24.80} = .153$$

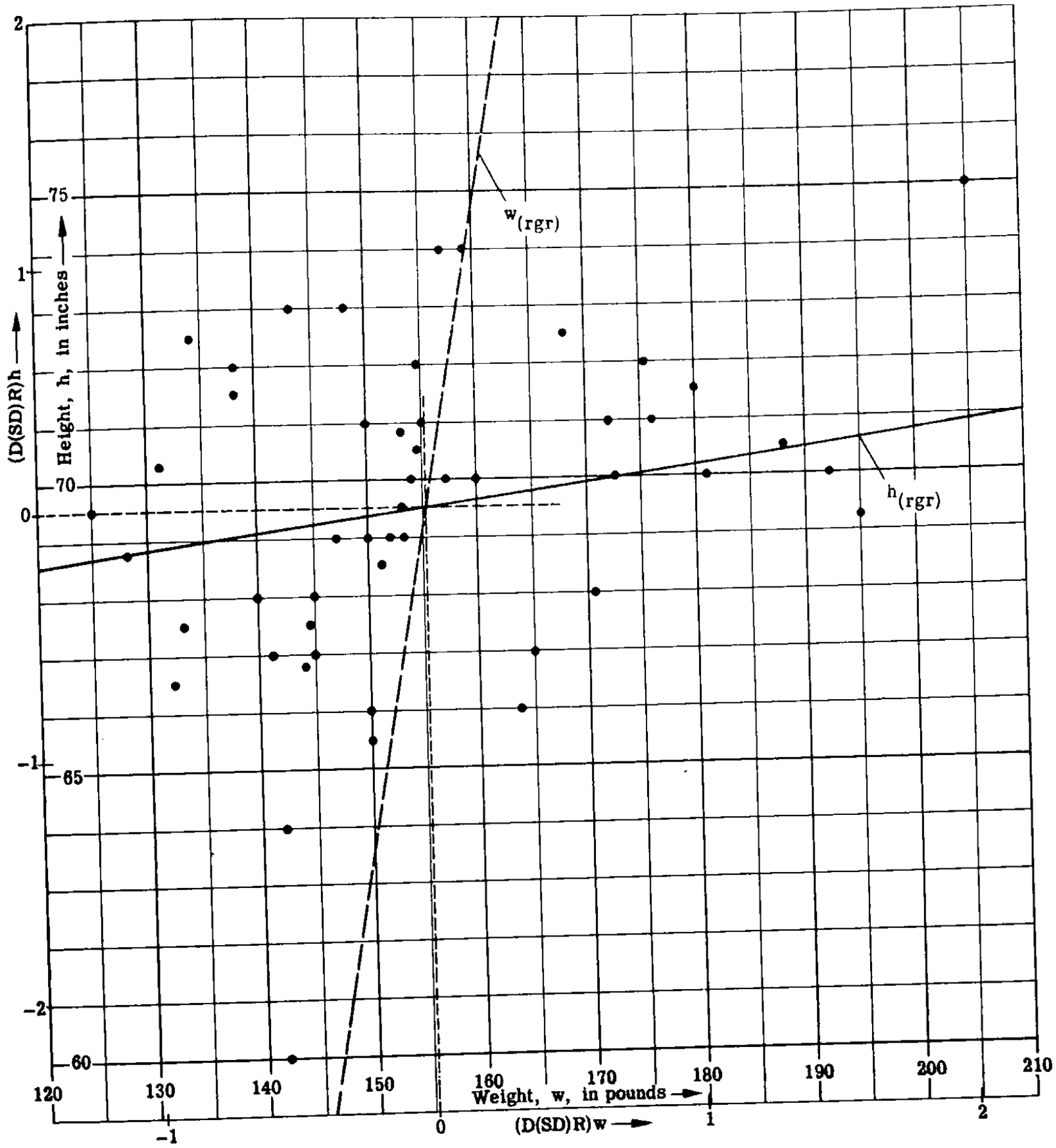


Fig. 11-9. Heights and weights of fifty male students at M.I.T.

Figure 11-9 shows the data from Table 11-5 presented in a "scatter plot," with the regression line of height on weight drawn in as a solid line. It is drawn through the average point, (Av)h = 69.505 in., (Av)w = 155.16 lb, with the computed slope of 0.0263 in./lb.

11.51. The regression method can be extended, if necessary, from a linear function of x to a power series, by fitting an x^2 curve to $[(D)y]_{(rgr)}$, then fitting an x^3 curve to the deviations from the x^2 curve, etc; but in most cases linear regression is adequate. Constructing the regression line of y on x implies that x is the independent variable and y is the dependent variable. It is also possible to reverse this implication by constructing the regression line of x on y. Both lines pass through the point [(Av)x, (Av)y]. The slope of the x on y regression line is

$$S_{(rgr)[y,x]} = \frac{(Av)[x_i (D)y_i]}{(Av)[(D)y_i]^2} \; . \qquad (11\text{-}47)$$

This slope is taken with respect to the y axis, while $S_{(rgr)[x;y]}$ is taken with respect to the x axis. The dashed line of Fig. 11-9 is the regression line of weight on height, drawn to a computed slope of 0.889 lb/in. Note that the numerators of the expressions for the two slopes are equal, and that the ratio of the two slopes is

$$\frac{S_{(rgr)[y;x]}}{S_{(rgr)[x;y]}} = \frac{(Av)[(D)x_i]^2}{(Av)[(D)y_i]^2} = \frac{[(SD)x]^2}{[(SD)y]^2} \; . \qquad (11\text{-}48)$$

These slopes must be either both positive or both negative, since the sign of both is determined by the sign of $(Av)[(D)x_i (D)y_i]$.

11.52. Another approach to correlation between two variables is by use of the correlation coefficient, defined as

$$(CC)[x,y] \equiv \frac{(Av)[(D)x_i (D)y_i]}{(SD)x \, (SD)y} \; . \qquad (11\text{-}49)$$

The correlation coefficient represents the fraction of the variations of either variable which depends on the variations of the other variable. Thus a correlation coefficient of +1 represents perfect positive proportionality; -1 represents perfect negative proportionality. Two independent variables would have a correlation coefficient of 0. It is evident that the correlation coefficient is the geometric mean of the two regression slopes,

$$(CC)[x,y] = \pm \sqrt{S_{(rgr)[y;x]} \, S_{(rgr)[x;y]}} \qquad (11\text{-}50)$$

(using the same sign as the regression slopes), and that it is the same regardless of which variable is considered independent and which dependent, i.e.,

$$(CC)[x,y] = (CC)[y,x] \; . \qquad (11\text{-}51)$$

11.53. For example, the correlation coefficient computed from the data of Table 11-5 is 0.153, meaning that 15.3 percent of the variations in either height or weight is attributable to the variations in the other. If the variables are expressed as their deviation — standard deviation ratios, the standard deviations become unity and the two regression slopes become equal to each other and to the correlation coefficient

$$S_{(rgr)}[(D(SD)R)x;\ (D(SD)R)y] = S_{(rgr)}[(D(SD)R)y;\ (D(SD)R)x] \tag{11-52a}$$

$$= (CC)[(D(SD)R)x,\ (D(SD)R)y] \tag{11-52b}$$

$$= (CC)[x,y] \tag{11-52c}$$

11.54. A correlation coefficient computed from n pairs of values contains an uncertainty, since a finite number of values is used. It can be shown that this uncertainty does not follow the normal probability law, but it is possible to introduce a new variable, z, the uncertainty of which approximately follows this law. Let $(CC)[x,y]_n$ be the value computed from n pairs of values, and let $(CC)[x,y]_\infty$ be the value which would be computed from an infinite number of values.

Define

$$z_n \equiv \frac{1}{2} \log_e \frac{1 + (CC)[x,y]_n}{1 - (CC)[x,y]_n} \tag{11-53a}$$

and

$$z_\infty \equiv \frac{1}{2} \log_e \frac{1 + (CC)[x,y]_\infty}{1 - (CC)[x,y]_\infty} \tag{11-53b}$$

Then z_n approximately follows the normal probability law, with the parameters

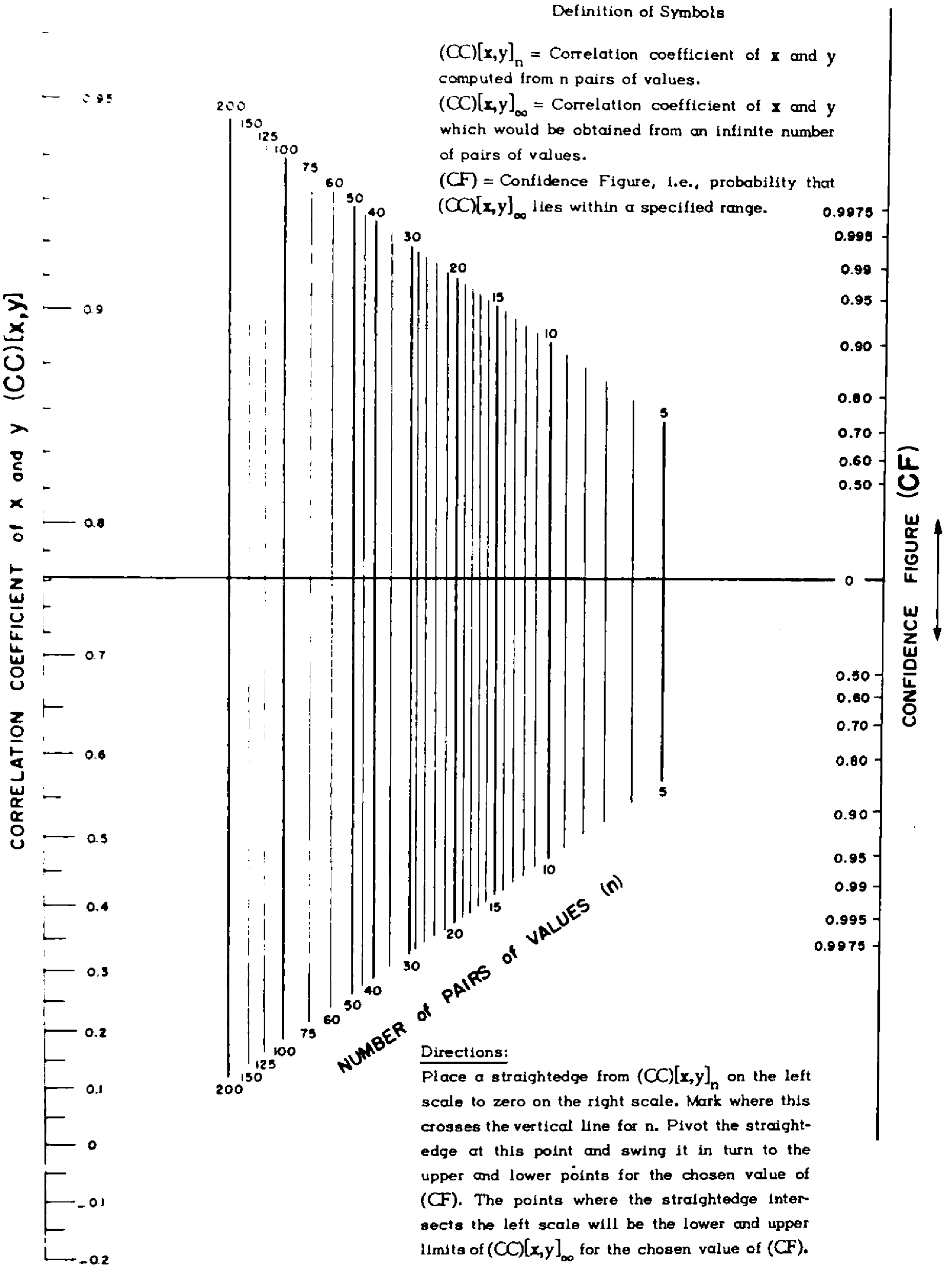
$$(Av)z_n = z_\infty \tag{11-54}$$

and

$$(SD)z_n = \frac{1}{\sqrt{n - 3}} \tag{11-55}$$

A nomogram for computing the uncertainty in $(CC)[x,y]_n$ is given in Fig. 11-10. This nomogram contains a nonlinear scale of (CC)[x,y] which effectively converts it to the normal probability variable, z.

11.55. Applying $(CC)[h,w]_n = 0.153$ and n = 50 to this nomogram shows that for a confidence figure of 0.95, $(CC)[h,w]_\infty$ could lie between + 0.41 and - 0.10. Since the usual experience with human height-weight correlation shows a coefficient of about 0.6, it is evident that this data is significantly different from the usual. Examination of the scatter plot of Fig. 11-9 shows a number of individuals in the range of less than 150 lb weight and more than 70 in. height. This combination of low weight with more than average height is immediately recognizable as unusual, and as the immediate cause of the low correlation in this example.



Definition of Symbols

$(CC)[x,y]_n$ = Correlation coefficient of x and y computed from n pairs of values.

$(CC)[x,y]_\infty$ = Correlation coefficient of x and y which would be obtained from an infinite number of pairs of values.

(CF) = Confidence Figure, i.e., probability that $(CC)[x,y]_\infty$ lies within a specified range.

Directions:

Place a straightedge from $(CC)[x,y]_n$ on the left scale to zero on the right scale. Mark where this crosses the vertical line for n. Pivot the straightedge at this point and swing it in turn to the upper and lower points for the chosen value of (CF). The points where the straightedge intersects the left scale will be the lower and upper limits of $(CC)[x,y]_\infty$ for the chosen value of (CF).

Example:

30 pairs of values give a correlation coefficient of .82; what are the 95% confidence limits for the correlation coefficient of an infinite number of pairs?
Answer: .65 and .91.

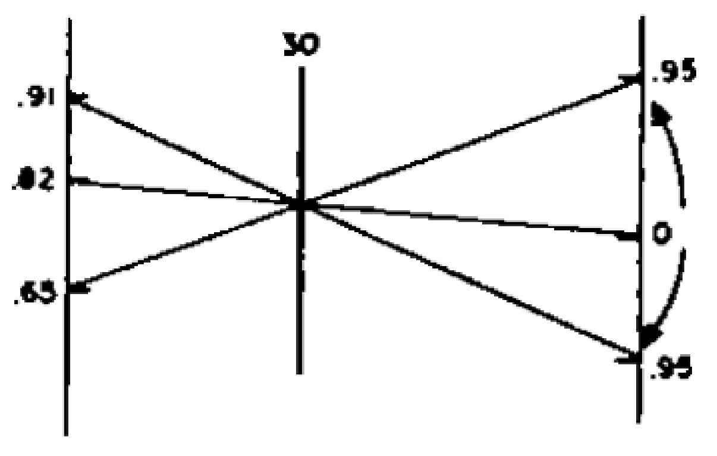


Fig. 11-10. Uncertainty in a correlation coefficient based on n pairs of values.

TIME SERIES AND AUTOCORRELATION

11.56. A time series is defined as any set of typical values of a variable such that changing the sequence of the values in the set would change some of the statistical properties of the set. A stationary time series is defined as a time series whose statistical properties (such as average value and standard deviation) remain constant throughout the series.

11.57. Now consider a stationary time series which is continuous, i.e., has a value at every instant of time, as for example an oscillograph recording of radio noise and signals, and consider the correlations between the values at two points a fixed time interval, T, apart. This interval is equivalent to a fixed distance between two curve tracers as an oscillogram record is moved past them. If T is small (i.e., if the curve tracers are close together) the curve will tend to have comparatively little difference in value between the two points, so that there will be some positive correlation between the value at any instant x_t, and the value at interval T later, $x_{(t+T)}$. This can be expressed as a correlation coefficient between x_t and $x_{(t+T)}$.

$$(CC)[x_t, x_{(t+T)}] = \frac{(Av)[x_t\, x_{(t+T)}]}{(SD)x_t\, (SD)x_{(t+T)}} \qquad (11\text{-}56)$$

If the record is continuous for x_t from initial time, $t_{(init)}$, to final time, $t_{(fin)}$, the average can be expressed as an integral

$$(Av)[x_t\, x_{(t+T)}] = \frac{1}{t_{(fin)} - t_{(init)}} \int_{t_{(init)}}^{t_{(fin)}} x_t\, x_{(t+T)}\, dt \qquad (11\text{-}57)$$

11.58. It is evident that T can be given any value, and the correlation coefficient will vary as a function of T. This function of T is called the autocorrelation function of x

$$(ACF)x \equiv (CC)[x_t, x_{(t+T)}] \Bigg|_{T = T_a}^{T = T_b} \qquad (11\text{-}58)$$

If x does not contain any periodic component (for example an oscillogram of thermal noise), (ACF)x will decrease exponentially as T increases. If x does contain a component periodic in $2\pi/\omega$ (for example a carrier superimposed on thermal noise), it is evident that when T is any integral multiple of $2\pi/\omega$, x_t and $x_{(t+T)}$ will tend to increase and decrease together, and hence (ACF)x will contain a component periodic in $2\pi/\omega$.

11.59. If the autocorrelation function is transformed from the time domain (a function of T) to the frequency domain (a function of ω) by means of the Fourier transform, it is found that the transformed function directly represents the frequency spectrum of the original time series. Since the product $x_t\, x_{(t + T)}$ has the dimensions of x^2 (equivalent to voltage squared in the electrical example), it is proportional to power rather than to voltage or its equivalent. These dimensions carry over into the transform of (ACF)x, so that the frequency spectrum is presented in terms of power. Hence, this transform is called the power spectral density of x, (PSD)x. Since (ACF)x is an "even" function of T (for a negative value of T it would have the same value as for the corresponding positive value of T) the Fourier transform takes the form

$$\begin{aligned}(\text{PSD})x &= (\text{FT})(\text{ACF})x \\ &= 2\int_0^{\infty} (\text{ACF})x \cos \omega T \, dT \qquad (11\text{-}59)\end{aligned}$$

11.60. The mathematical theory on which (ACF)x and (PSD)x are founded calls for integration to infinity, using a time series from $t = -\infty$ to $t = +\infty$, and all values of T from $T = 0$ to $T = +\infty$. In practical applications it is sufficient to use closely-spaced values of T over the range in which any appreciable energy is to be expected, and to use a record with a duration many times as great as the largest value of T, i.e.,

$$[t_{(\text{fin})} - t_{(\text{init})}] \gg T_{(\text{max})} \qquad (11\text{-}60)$$

The difference between the theoretical time series of infinite duration and the practical time series of limited duration results in the practical (PSD)x curve having a narrow peak at the frequency of each sinusoidal component, where the theoretical curve would have a "delta function," a peak of zero width, infinite height, and finite area. A time series consisting of a finite band of frequencies (for example a speech signal) will yield a (PSD)x curve which describes the relative power density through the band. In the practical case of a series of limited duration, the band will be very slightly broadened and any very steep slopes will be slightly flattened.

CHAPTER 12

REPRESENTATION OF PHYSICAL SITUATIONS BY MATHEMATICAL FORMS

PHYSICAL QUANTITIES AS FUNCTIONS OF A RUNNING VARIABLE

12.1. Representation of physical quantities by mathematical forms is a primary requirement if the processes of theoretical analysis are to be usefully applied to the problems of instrument engineering. This representation must provide both for variations that are free from roughness and for variations that contain uncertainty or roughness components of appreciable magnitude.

12.2. The statistical methods discussed in the preceding chapter will usually be required to represent uncertainty and roughness components.* It is the object of the present chapter to describe generally useful forms for representing changes in which roughness effects may be disregarded. This means that consideration is limited to changes in theoretical or actual variations that remain when roughness components are removed by the action of some system, or are disregarded for the purposes of discussion. The quantity-describing forms must be suitable for representing variations with respect to time of the independent quantities that act as the inputs for operating components or systems, and also the dependent quantities that are the corresponding outputs. Because the dimensions of the quantity described do not affect the mathematical form of the describing function required in any given case, simplified working variables are generally used to represent actual quantities. The working variable symbol is $q(t)$, to show that the quantity described is considered as a function of time or any other running variable that may be represented by t.

12.3. The material of this chapter is not intended to give rigorous proofs for the mathematical properties of the forms used to represent physical variations. These proofs are generally available in texts on mathematics(10),(17),(21) and are not required for present purposes. The information required to apply the simple power series, the impulse function, the Fourier series, and the Fourier integral to describe variations of the working variable as a function of the running variable is summarized from the standpoint of engineering applications. Dimensionless ratios are introduced to generalize representations of essential variations and to stress the importance of working in terms of these ratios. The various performance-describing forms and the auxiliary information necessary for engineering applications are summarized in a series of definition summaries and figures designed for ready reference by engineers who are concerned with practical applications rather than with the niceties of mathematical derivations.

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* Exceptions to this rule may occur in problems involving roughness. For example, it may sometimes be possible to approximate roughness components by means of a combination of sinusoidal terms.

POWER SERIES REPRESENTATION

12.4. It is a well-known theorem of mathematics(3),(7),(36) that variations of the kind generally associated with physical quantities may be expressed in terms of a power series. The typical power series form is shown in Eq. (1) of Definition Summary 12-1. This equation relates q(t) to the sum of a constant term, B_0, and all powers of t, each multiplied by a coefficient B_1, B_2, ... The individual terms of the power series may be given special names and represented as the ordinate, with t as the abscissa.

Step Function — For example, Definition Summary 12-1b illustrates the effect of a single, constant term B_0 that appears suddenly with a positive value at the initial instant. A change of this kind has the profile of an upward step and, for this reason, is called the increasing step function. To illustrate the general method for describing the situation represented by an equation, Definition Summary 12-1b gives a summary that specifies the increasing step function in concise mathematical terms. Definition Summary 12-1c is a description of the decreasing step function, which corresponds to a power series with all coefficients equal to zero except for the constant term, which is assumed to change instantaneously from B_0 to 0 at the initial instant.

Ramp Function — Sections d) and e) of Definition Summary 12-1 represent the variations corresponding to the first-power term of the power series when the coefficient is respectively positive and negative. Graphically represented, this term corresponds to a straight line sloping upward from zero when the first-power coefficient is positive. This variation is called the increasing ramp function because it has a shape like the profile of a ramp. The decreasing ramp function appears when the coefficient B_0 is positive, and the first-power coefficient B_1 comes suddenly into existence with a negative value at the initial instant t = 0.

Parabolic Function — Sections f) and g) of Definition Summary 12-1 illustrate respectively the increasing parabolic function and the decreasing parabolic function, which correspond to the positive and negative values of the second-power coefficient B_2.

The generalized infinite power series in the running variable t has the form

$$q(t) = B_0 + B_1t + B_2t^2 + B_3t^3 + B_nt^n \dots \qquad (1)$$

where

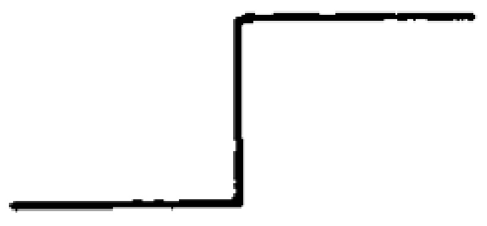
$B_0, B_1 \dots B_n$ are constants

t = running variable to be determined for each different $q(t)$.

a) Representation of a physical quantity by a power series

A step function exists if $B_1 = B_2 = B_3 \dots = 0$

and

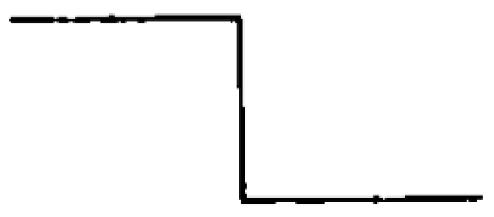


$t = 0$ (initial instant)

$B_0 = 0$	$t < 0$
$B_0 = 0 +$ (constant)	$t \leqq 0$
$q(t) = 0$	$t < 0$
$q(t) = B_0$	$t \leqq 0$

b) Increasing step function

and



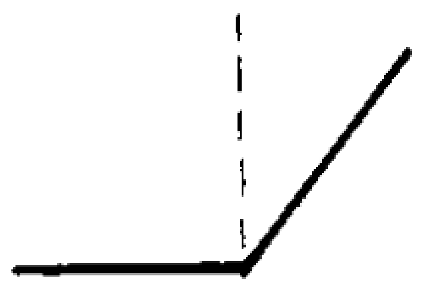
$t = 0$

$B_0 = 0 +$ (constant)	$t < 0$
$B_0 = 0$	$t \leqq 0$
$q(t) = B_0$	$t < 0$
$q(t) = 0$	$t \leqq 0$

c) Decreasing step function

A ramp function exists if $B_2 = B_3 = B_4 \dots = 0$

and

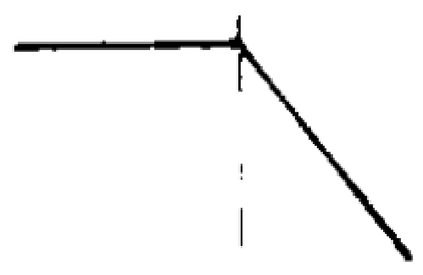


$t = 0$

$B_0 = 0$	$t = 0$
$B_1 = 0$	$t < 0$
$B_1 = +$ (constant)	$t \geqq 0$
$q(t) = 0$	$t < 0$
$q(t) = B_1t$	$t \leqq 0$

d) Increasing ramp function

and



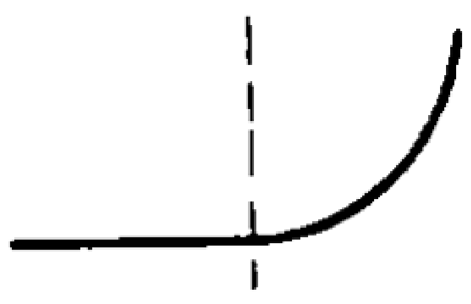
$t = 0$

$B_0 = +$ (constant)	$t = 0$
$B_1 = 0$	$t < 0$
$B_1 = -$ (constant)	$t \leqq 0$
$q(t) = B_0$	$t < 0$
$q(t) = B_0 - B_1t$	$t \leqq 0$

e) Decreasing ramp function

A parabolic function exists if $B_1 = B_3 = B_4 = B_5 \dots = 0$

and

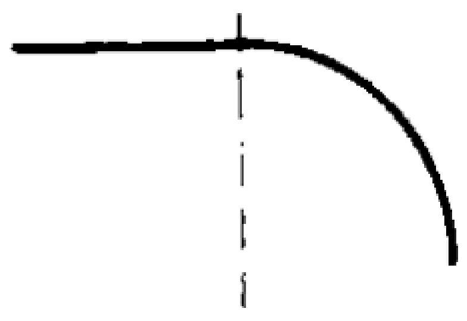


$t = 0$

$B_0 = 0$	
$B_2 = 0$	$t < 0$
$B_2 = +$ (constant)	$t \geqq 0$
$q(t) = 0$	$t < 0$
$q(t) = B_2t^2$	$t \geqq 0$

f) Increasing parabolic function

and



$t = 0$

$B_0 = +$ (constant)	
$B_2 = 0$	$t < 0$
$B_2 = -$ (constant)	$t \leqq 0$
$q(t) = B_0$	$t < 0$
$q(t) = B_0 - B_2t^2$	$t \leqq 0$

g) Decreasing parabolic function

Definition Summary 12-1. Representation of a physical quantity by a generalized power series.

All the terms of the power series may be represented individually by the method illustrated for the constant term, the first-power term, and the second-power term. When more than one is present, the graph representing the working variable is the resultant of all the curves for the individual terms. The constant term, the first-power term, and the second-power term are forms that have engineering applications. A number of these applications are discussed in Volume III. When more than the first three power series terms are required to describe a physical situation, the power series becomes so cumbersome from the standpoint of mathematical treatment that it is generally advantageous to abandon this form for functions better adapted to the representation of complicated variations. It is also generally true that the work of finding the solution for a given differential equation (like the performance equation derived in Chapter 8 for the spring and piston engine indicator) when the independent variable is expressed as a power series is complicated by the fact that each term of the series is different. Except for the especially simple situation mentioned above, it is usually good procedure to represent quantities by means of functions with component terms that, from the standpoint of the operations involved in solving differential equations, may be described by a single mathematical form.

EXPONENTIAL SERIES

12.5. A procedure often applied to improve the usefulness of the power series method is to change the basic form of Eq. (1), Definition Summary 12-1, into another series with each term in itself an infinite power series. These component series are defined in many ways for convenience in fitting the requirements of special situations. One of the most widely effective of the possible component forms is the exponential series,* defined in Eq. (1) of Definition Summary 12-2. The special advantage of this series for differential equation work is that it is unchanged by differentiation with respect to its running variable (t). Because the immediate concern is in representing quantity variations rather than in finding formal solutions for differential equations, no attention is given here to the mathematical behavior of the exponential series. This behavior is discussed at length and is applied many times in finding solutions for the formal equations that are considered in Volume II.

12.6. Section a) of Definition Summary 12-2 shows that the exponential form is written as the base of natural logarithms, e, raised to the kt power. In this exponent the coefficient k must have the dimension of inverse time in order for the exponent itself to fulfill the basic requirement of no dimensions. The exponential series is defined as the sum of unity and a series of terms including all

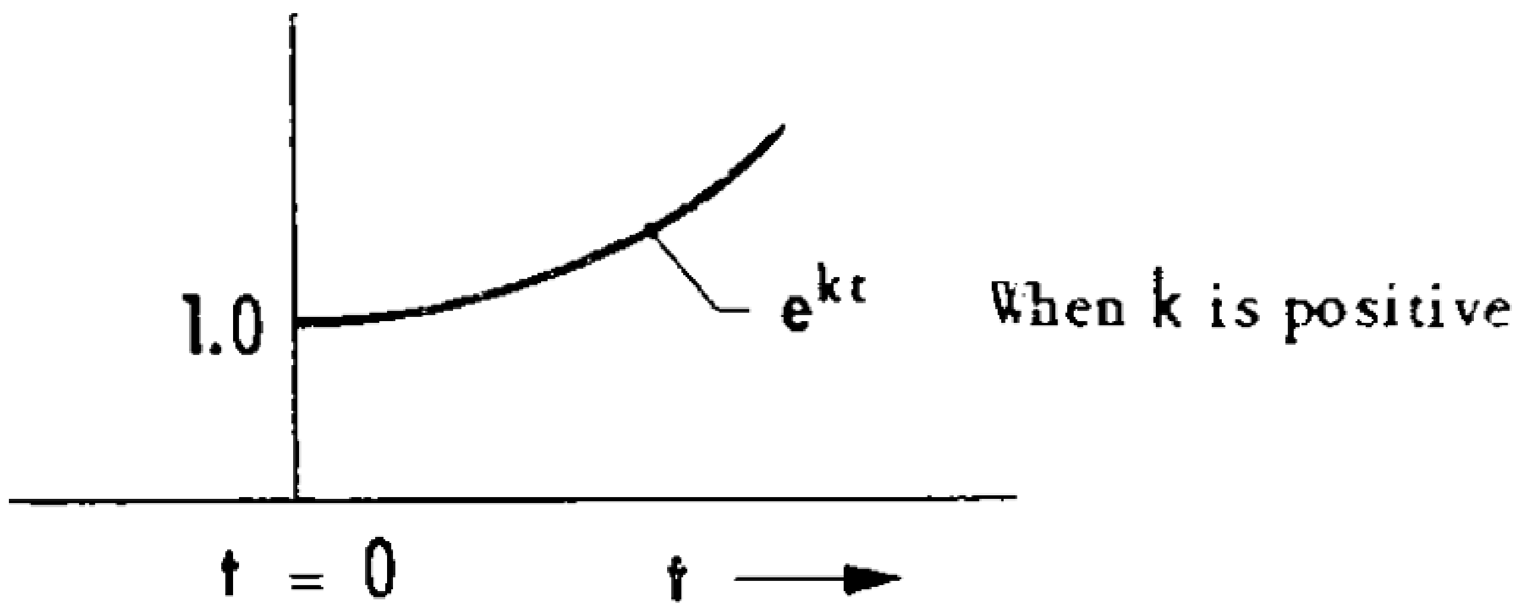
---

* The exponential series is defined and its important properties are discussed by Franklin in Methods of Advanced Calculus (17).

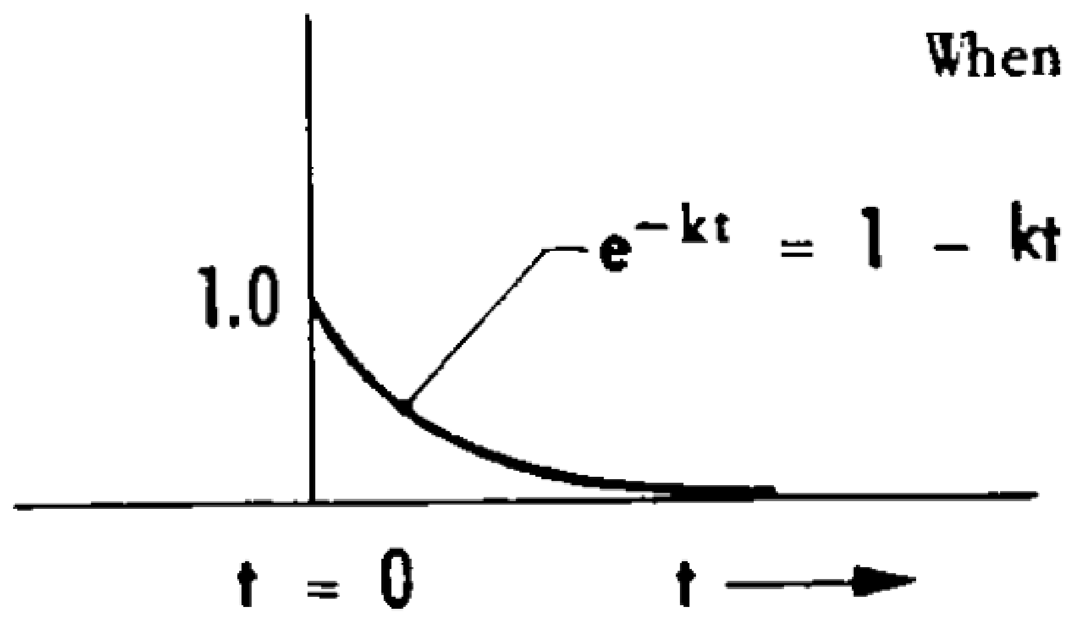
The generalized power series in t may be written as a sum of specified power series forms. For example, the exponential series is defined as the sum of an infinite series of terms with the form of

$$e^{kt} = 1 + kt + \frac{(kt)^2}{2!} + \frac{(kt)^3}{3!} + \ldots \tag{1}$$

a) Definition of exponential series



b) Geometrical representation of positive exponential.



When k is negative

$$e^{-kt} = 1 - kt + \frac{(kt)^2}{2!} - \frac{(kt)^3}{3!} + \ldots \tag{2}$$

c) Geometrical representation of negative exponential.

The $q(t)$ generalized power series form of Eq. (1), Definition Summary 12-1, might be written

$$\begin{aligned} q(t) = B'_0 &+ B'_{1(+)} e^{+k_1 t} + B'_{2(+)} e^{+2k_1 t} + B'_{3(+)} e^{+3k_1 t} + \ldots \\ &+ B'_{1(-)} e^{-k_1 t} + B'_{2(-)} e^{-2k_1 t} + B'_{3(-)} e^{-3k_1 t} + \ldots \end{aligned} \tag{3}$$

where values of B'_0, $B'_{1(+)}$, $B'_{1(-)}$ . . . must be determined to fit any given function $q(t)$.

Geometrically, this equation means that when k is real, the quantity $q(t)$ would be represented as the resultant of an infinite series of component curves like those of b) and c). The situation that exists when k is imaginary is discussed in Definition Summary 12-3, and illustrated in Fig. 12-1.

d) Representation by exponential series.

Definition Summary 12-2. The generalized power series in terms of exponential series components.

powers of kt, each term having a coefficient equal to the reciprocal of the factorial for the power of kt appearing in its exponent. The series definition applies for values of k that may be either real or complex, and positive or negative. Typical exponential curves for real values of positive and negative k are shown in sections c) and d) of Definition Summary 12-2. The positive exponent gives a curve that starts at unity and diverges from the zero axis of the plot.

The negative exponent gives a curve that starts at unity and asymptotically approaches the zero axis with a slope that is proportional to the ordinate of the plot. In practice, the positive exponential is a typical form for representing instability of a function while the negative exponential is a typical form for representing stability of a function.

SERIES OF EXPONENTIALS

12.7. Section d) of Definition Summary 12-2 illustrates the form of a series for q(t) made up of a constant and an infinite set of terms, each of which is itself an exponential with an integer corresponding to the order of each term multiplying the positive k of the exponent. A similar set of terms occurs for negative values of k. Each exponential term is multiplied by a coefficient that must be given a proper value when the series is used to describe a given function q(t). When the value chosen for k_1 is real, the series of Eq. (3) of Definition Summary 12-2 represents the sum of an infinite set of positive exponentials and an infinite set of negative exponentials, each of which corresponds to some curve shape like the plots of sections b) and c) of Definition Summary 12-2. It is possible to cause the resultant of these exponential component terms to represent a given function q(t) with the same degree of satisfaction that the simple power series of Definition Summary 12-1 could be made to represent the given function. From the standpoint of analysis, it is advantageous to use the exponential series form in describing functions of t because a formal solution for a typical exponential term may be easily applied to all the exponential terms.

IMAGINARY EXPONENTIALS

12.8. An especially useful form of the exponential series appears when the k of the defining equation becomes imaginary. For the purposes of discussion k will be replaced by the symbol jw, where j represents the square root of minus one, and w is a coefficient with the dimension of inverse time. It is shown later in the discussion of exponentials that w corresponds to the revolution frequency of a rotating vector that may be associated with the imaginary exponential function. For this reason it is called the angular frequency, which is equal to 2π times the frequency, n. Equations (1) and (2) of Definition Summary 12-3 are the forms taken by the exponential series when the exponent is respectively positive imaginary and negative imaginary.

RELATIONSHIPS OF EXPONENTIALS AND TRIGONOMETRIC FUNCTIONS

12.9. The graphical interpretation of the imaginary exponential series is simplified by noting that the real part of the exponential is a series identical with the defining series for cos wt, and the imaginary part of the exponential series

The exponential series may be written with the coefficient k imaginary instead of real as in the defining Eq. (1) of Definition Summary 12-2. The imaginary exponential series appears if $\pm k$ becomes $\pm jw$ where

$j = \sqrt{-1}$ = imaginary unit
$w = 2\pi n$ = angular frequency
n = frequency

The corresponding exponential series forms are

$$e^{jwt} = 1 - \frac{(wt)^2}{2!} + \frac{(wt)^4}{4!} - \frac{(wt)^6}{6!} + \dots$$

$$+ j\left[wt - \frac{(wt)^3}{3!} + \frac{(wt)^5}{5!} - \dots\right] \tag{1}$$

$$e^{-jwt} = 1 - \frac{(wt)^2}{2!} + \frac{(wt)^4}{4!} - \frac{(wt)^6}{6!} + \dots$$

$$- j\left[wt - \frac{(wt)^3}{3!} + \frac{(wt)^5}{5!} + \dots\right] \tag{2}$$

a) Definition of imaginary exponential series

It is useful to note that the real and imaginary component series of Eq. (2) are the equations for the trigonometric sine and cosine series, i.e.,

$$\cos wt = 1 - \frac{(wt)^2}{2!} + \frac{(wt)^4}{4!} - \frac{(wt)^6}{6!} + \dots \tag{3}$$

$$\sin wt = wt - \frac{(wt)^3}{3!} + \frac{(wt)^5}{5!} - \frac{(wt)^7}{7!} + \dots \tag{4}$$

When cosine and sine terms are used to replace the component series in Eq. (1), the resulting equations are

$$e^{jwt} = \cos wt + j \sin wt \tag{5}$$

$$e^{-jwt} = \cos wt - j \sin wt \tag{6}$$

The solution of these equations for the trigonometric functions in terms of the exponential shows that

$$\cos wt = \frac{e^{jwt} + e^{-jwt}}{2} \tag{7}$$

$$\sin wt = \frac{e^{jwt} - e^{-jwt}}{2j} \tag{8}$$

b) Relationships between imaginary exponentials and trigonometric functions

Definition Summary 12-3. Relationships among exponential functions and trigonometric functions.

is identical with the defining series for sin wt. These trigonometric series forms are given by Eqs. (3) and (4) of Definition Summary 12-3. The essential relationships among the trigonometric functions, and positive and negative imaginary exponentials, are summarized in Eqs. (5), (6), (7) and (8) of Definition Summary 12-3.

12.10. Figure 12-1 summarizes the geometrical interpretation of the imaginary exponential form when the argument, (wt), is a real constant and the exponential, IE, (for imaginary exponential), is multiplied by a real constant $(IE)_a$. This constant is the amplitude of (IE). The magnitude of the exponential e^{jwt} is proportional to the square root of the sum of the squares of cos $(wt)_1$ and sin $(wt)_1$. Because the sum of sine and cosine squares is always unity, it follows that the magnitude of the exponential itself is always unity. This statement is true for both positive and negative values of the imaginary exponents.

Geometrical interpretation of the imaginary exponentials is based on the relationships

$$(IE) = (IE)_a e^{jwt} = (IE)_a[\cos(wt)_1 + j\sin(wt)_1] \tag{1}$$

and

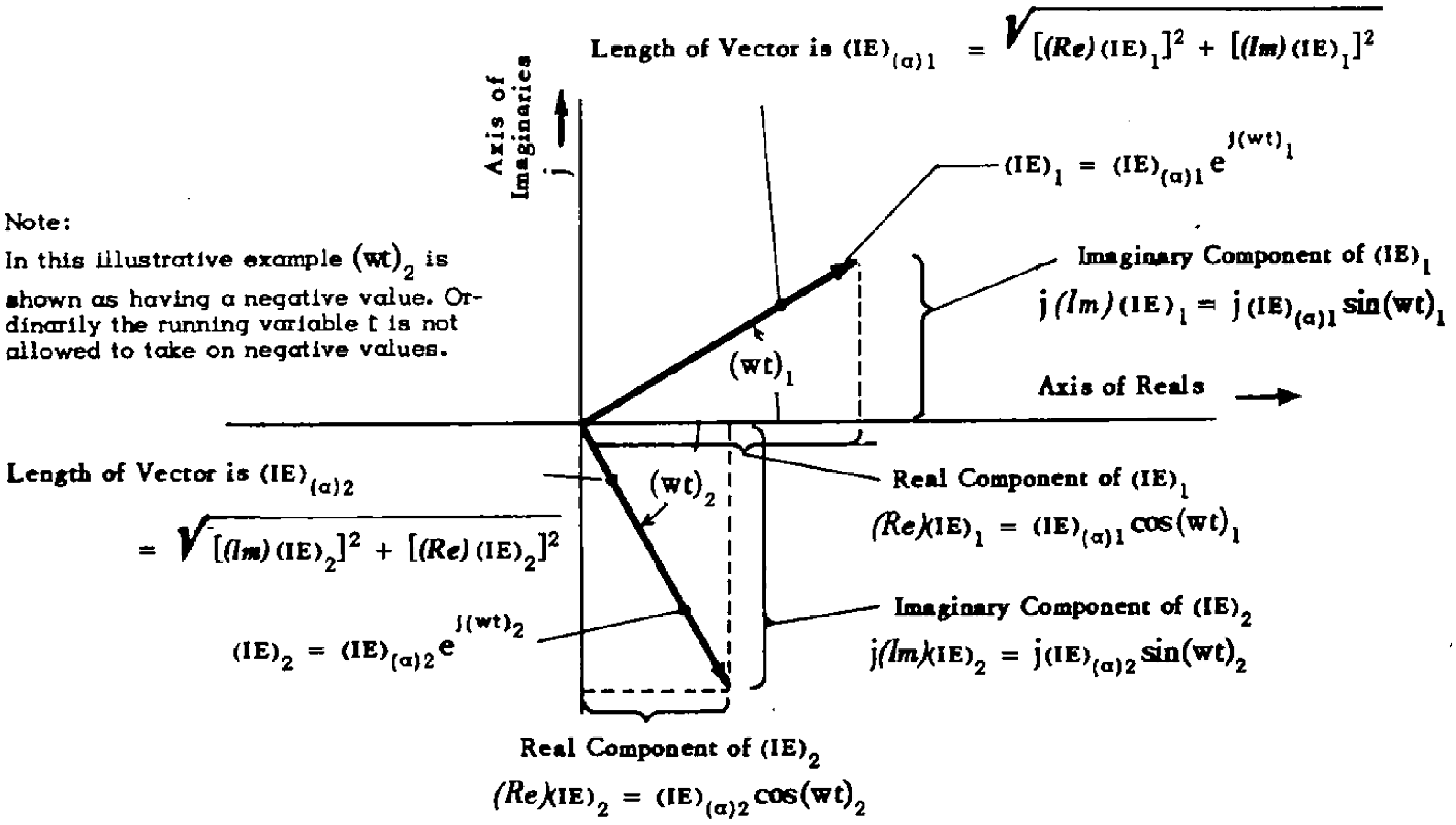
$$(IE) = (IE)_a e^{-jwt} = (IE)_a[\cos(wt)_2 - j\sin(wt)_2] \tag{2}$$

where

$(IE)_a$ = amplitude of imaginary exponential

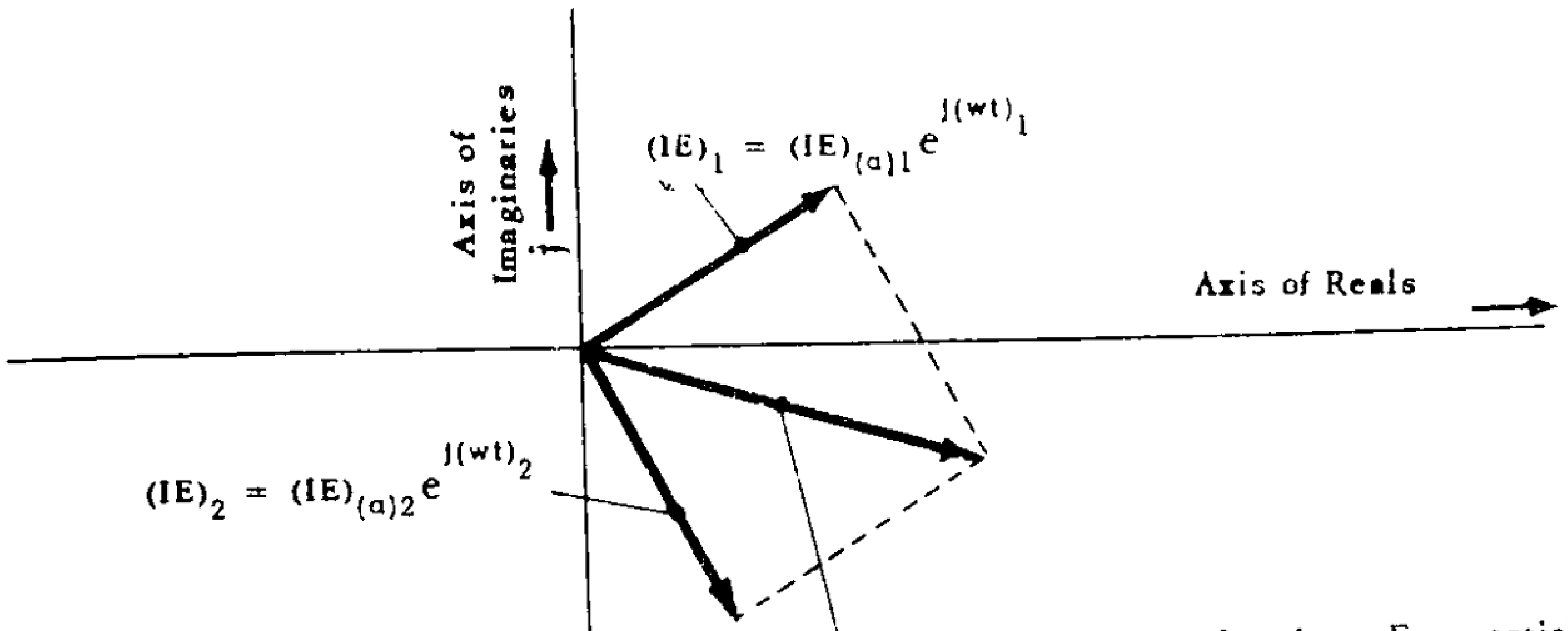
(wt) = argument of imaginary exponential

When the positive and negative exponentials have arbitrary constant amplitudes and constant arguments, $(IE)_{(a)1}$, $(IE)_{(a)2}$, and $(wt)_1$, $(wt)_2$, respectively, the situation may be represented by vectors plotted on the complex plane; i.e.,


Length of Vector is $(IE)_{(a)1} = \sqrt{[(Re)(IE)_1]^2 + [(Im)(IE)_1]^2}$
Axis of Imaginaries
j
$(IE)_1 = (IE)_{(a)1}e^{j(wt)_1}$
Note:
In this illustrative example $(wt)_2$ is shown as having a negative value. Ordinarily the running variable t is not allowed to take on negative values.
Imaginary Component of $(IE)_1$
$j(Im)(IE)_1 = j(IE)_{(a)1}\sin(wt)_1$
$(wt)_1$
Axis of Reals
Length of Vector is $(IE)_{(a)2}$
$= \sqrt{[(Im)(IE)_2]^2 + [(Re)(IE)_2]^2}$
$(wt)_2$
Real Component of $(IE)_1$
$(Re)(IE)_1 = (IE)_{(a)1}\cos(wt)_1$
$(IE)_2 = (IE)_{(a)2}e^{j(wt)_2}$
Imaginary Component of $(IE)_2$
$j(Im)(IE)_2 = j(IE)_{(a)2}\sin(wt)_2$
Real Component of $(IE)_2$
$(Re)(IE)_2 = (IE)_{(a)2}\cos(wt)_2$

a) Graphical representation of imaginary exponentials with constant arguments

Fig. 12-1. Graphical representation of addition and subtraction for imaginary exponentials. (Page 1 of 2)



$(IE)_1 = (IE)_{(a)1} e^{j(wt)_1}$

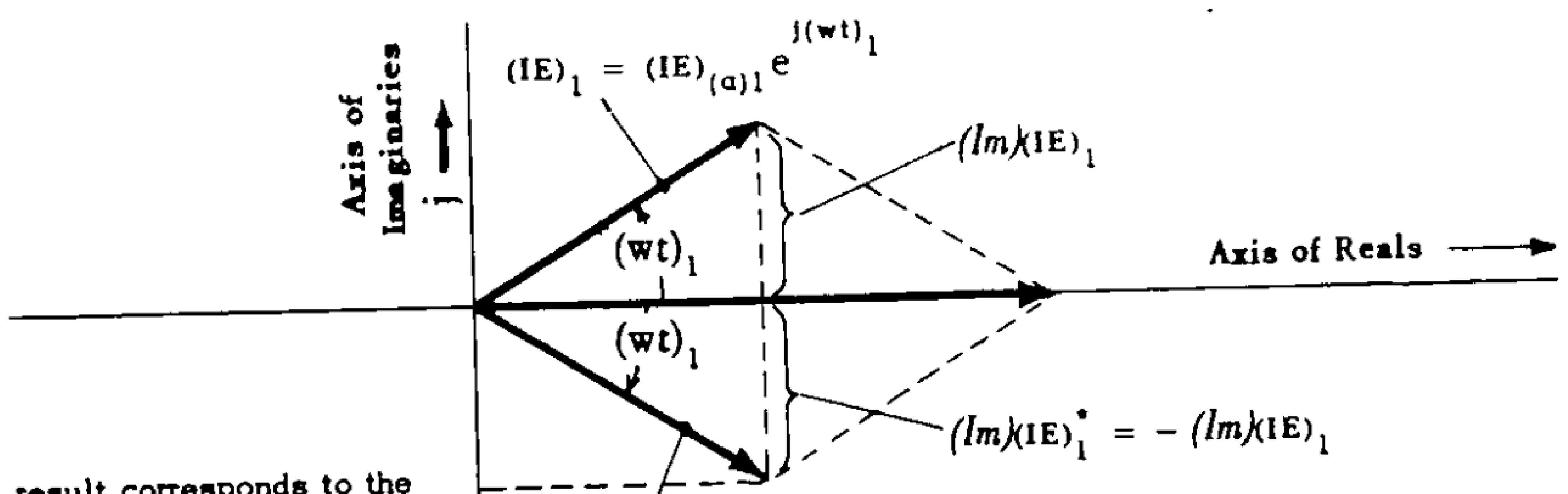
$(IE)_2 = (IE)_{(a)2} e^{j(wt)_2}$

Subtraction is carried out as addition, with the sign reversed on the term to be subtracted.

$(IE)_{res} = (IE)_1 + (IE)_2 =$ Imaginary Exponential Resultant
$= (Re)(IE)_1 + (Re)(IE)_2 + j[(Im)(IE)_1 + (Im)(IE)_2]$

Length of $(IE)_{res} = \sqrt{(IE)_1^2 + (IE)_2^2}$

b) Graphical representation of the sum of imaginary exponentials



$(IE)_1 = (IE)_{(a)1} e^{j(wt)_1}$

$(Im)(IE)_1$

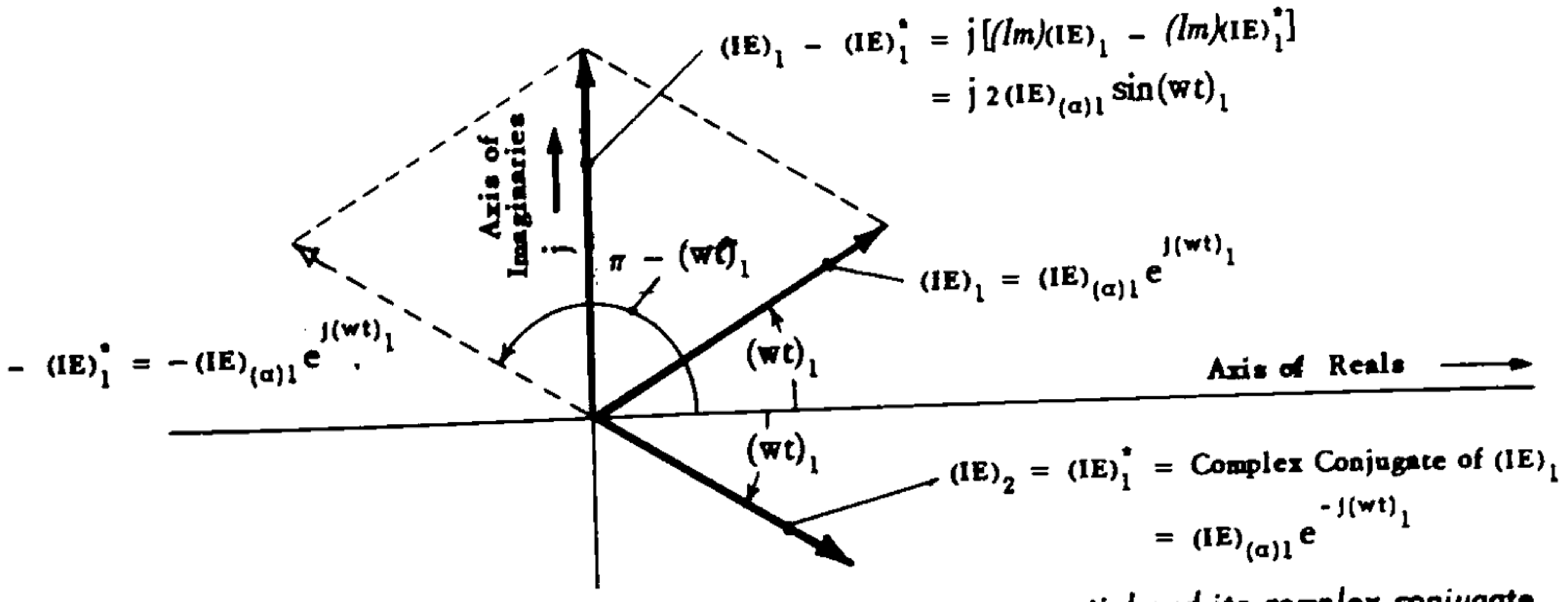
$(Im)(IE)_1^* = -(Im)(IE)_1$

Note:
This graphical result corresponds to the relationship of Eq. (7) of Definition Summary 12-3.

$(IE)_2 = (IE)_1^* =$ Complex Conjugate of $(IE)_1$
$= (IE)_{(a)1} e^{-j(wt)_1}$

$$(IE)_1 + (IE)_1^* = (Re)(IE)_1 + (Re)(IE)_1^* + j[(Im)(IE)_1 + (Im)(IE)_1^*]$$
$$= 2(Re)(IE)_1 = 2(IE)_{(a)1}\cos(wt)_1$$

c) Graphical representation of the sum of an imaginary exponential and its complex conjugate



$(IE)_1 - (IE)_1^* = j[(Im)(IE)_1 - (Im)(IE)_1^*]$
$= j\,2(IE)_{(a)1}\sin(wt)_1$

$(IE)_1 = (IE)_{(a)1} e^{j(wt)_1}$

$-(IE)_1^* = -(IE)_{(a)1} e^{j(wt)_1}$

$(IE)_2 = (IE)_1^* =$ Complex Conjugate of $(IE)_1$
$= (IE)_{(a)1} e^{-j(wt)_1}$

d) Graphical representation of the difference of an imaginary exponential and its complex conjugate

Fig. 12-1. Graphical representation of addition and subtraction for imaginary exponentials. (Page 2 of 2)

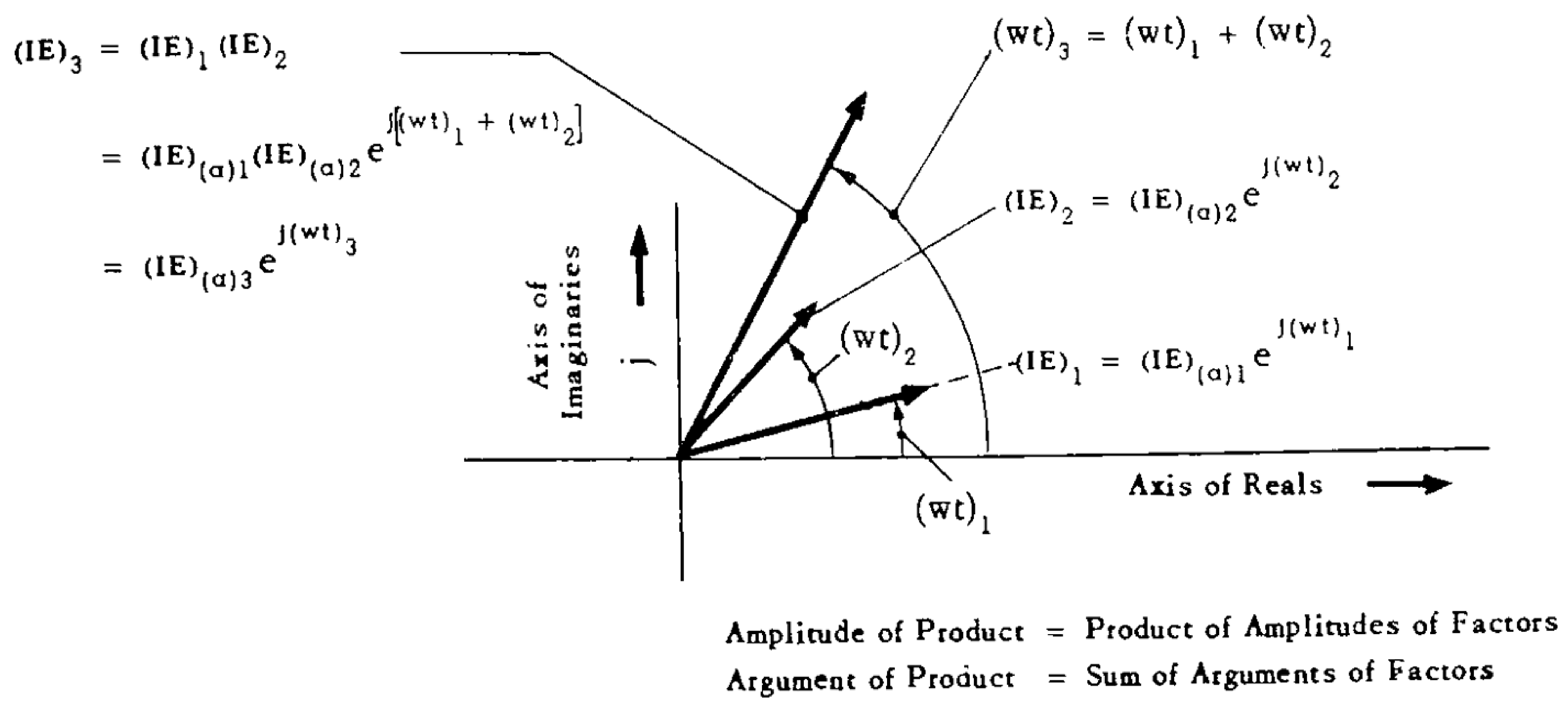
COMPLEX PLANE REPRESENTATION OF IMAGINARY EXPONENTIALS

12.11. Figure 12-1a shows the representation of imaginary exponentials on the complex plane, which has reals plotted as abscissae and imaginaries (terms that include j as a factor) as ordinates. One of the two illustrative exponentials shown in this figure has the constant argument $(wt)_1$. The real component of this term (represented by $(Re)(IE)_1$) is along the axis of reals and is equal to the amplitude multiplied by the cosine of the argument. The corresponding imaginary component of $(IE)_1$ falls along the imaginary axis and is equal to the amplitude multiplied by the sine of the argument. The imaginary exponential itself is completely represented by a vector of length $(IE)_{(a)1}$ making an angle $(wt)_1$ measured counterclockwise from the positive direction of the axis of reals. The corresponding representation for $(IE)_2$ is a vector of length $(IE)_{(a)2}$ making a negative angle $(wt)_2$ (i.e., measured clockwise because $(wt)_2$ has been taken as negative in the illustrative example) from the positive direction of the real axis. In terms of real and imaginary components, the amplitude of any given imaginary exponential is equal to the square root of the sum of the squares of the real and imaginary components.

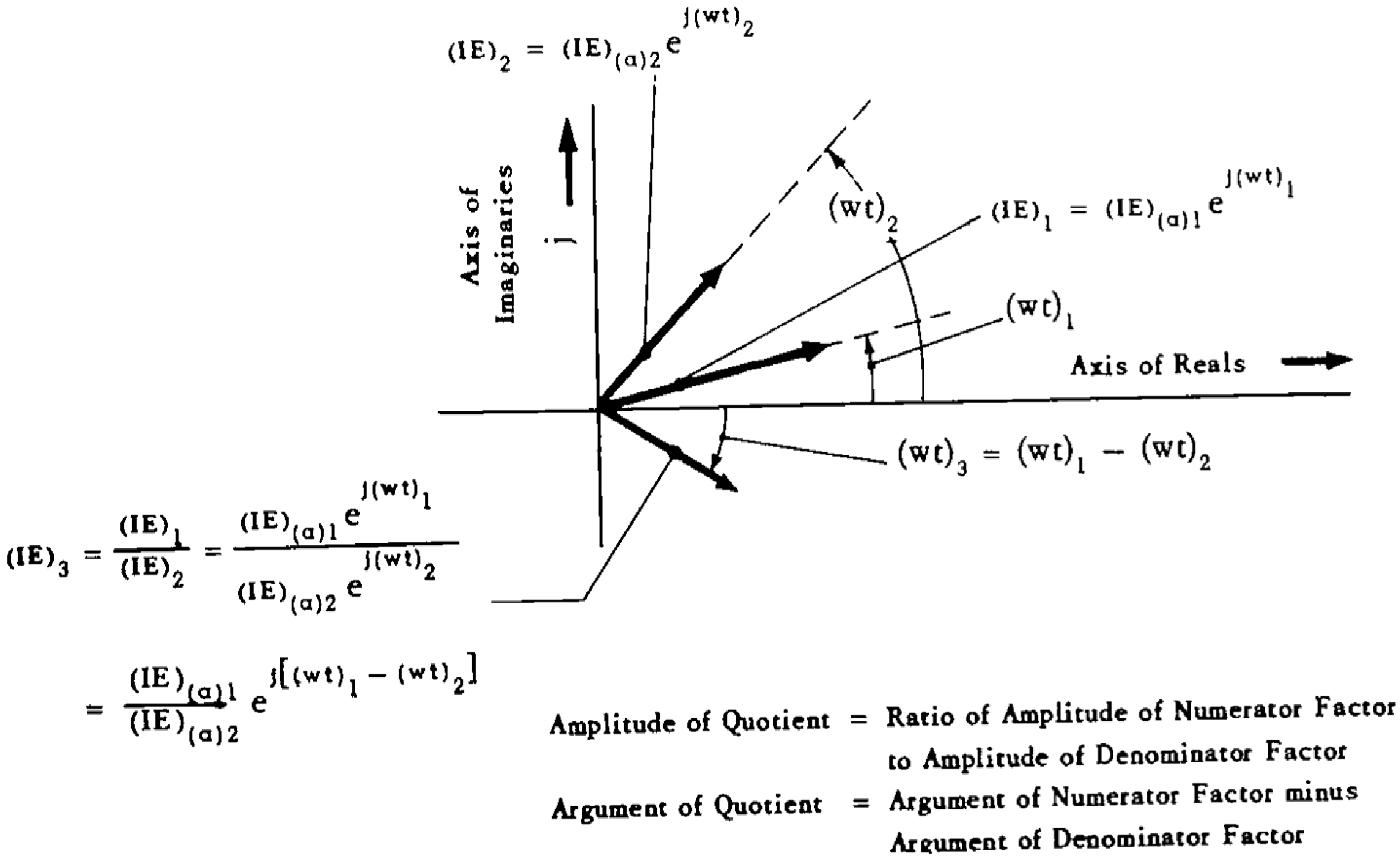
ADDITION OF IMAGINARY EXPONENTIALS

12.12. Figure 12-1b illustrates the rule for addition of imaginary exponentials. Graphically, the sum of two exponentials is the resultant of the two vectors that represent the individual exponentials. The real component of the resultant is equal to the sum of the real components of the exponential terms. Similarly, the imaginary component of the resultant is equal to the sum of the imaginary components of the exponential terms. The subtraction of two exponentials becomes identical with addition when the sign of the term to be subtracted is reversed.

12.13. A special case of addition occurs when an exponential and its complex conjugate are combined. By definition, the complex conjugate of a given exponential is an exponential with its amplitude equal to the amplitude of the given exponential and with its argument equal to the negative of the argument of the given exponential. In effect, this definition means that the real component of an imaginary exponential and its complex conjugate are equal, while the imaginary components have equal magnitudes but opposite signs. Figure 12-1c shows that the sum of an imaginary exponential and its complex conjugate is a real quantity equal to twice the real component of the exponential. The corresponding difference of a given imaginary exponent and its complex conjugate is an imaginary quantity equal to twice the imaginary component of the given exponential. This difference is illustrated by the diagram of Fig. 12-1d.


$(IE)_3 = (IE)_1 (IE)_2$
$= (IE)_{(a)1}(IE)_{(a)2}e^{j[(wt)_1 + (wt)_2]}$
$= (IE)_{(a)3}e^{j(wt)_3}$
$(wt)_3 = (wt)_1 + (wt)_2$
$(IE)_2 = (IE)_{(a)2}e^{j(wt)_2}$
$(wt)_2$
$(IE)_1 = (IE)_{(a)1}e^{j(wt)_1}$
Axis of Imaginaries
j
Axis of Reals
$(wt)_1$
Amplitude of Product = Product of Amplitudes of Factors
Argument of Product = Sum of Arguments of Factors

a) Graphical representation for product of two imaginary exponentials


$(IE)_2 = (IE)_{(a)2}e^{j(wt)_2}$
Axis of Imaginaries
j
$(wt)_2$
$(IE)_1 = (IE)_{(a)1}e^{j(wt)_1}$
$(wt)_1$
Axis of Reals
$(wt)_3 = (wt)_1 - (wt)_2$
$(IE)_3 = \frac{(IE)_1}{(IE)_2} = \frac{(IE)_{(a)1}e^{j(wt)_1}}{(IE)_{(a)2}e^{j(wt)_2}}$
$= \frac{(IE)_{(a)1}}{(IE)_{(a)2}}e^{j[(wt)_1 - (wt)_2]}$
Amplitude of Quotient = Ratio of Amplitude of Numerator Factor to Amplitude of Denominator Factor
Argument of Quotient = Argument of Numerator Factor minus Argument of Denominator Factor

b) Graphical representation for division of two imaginary exponentials

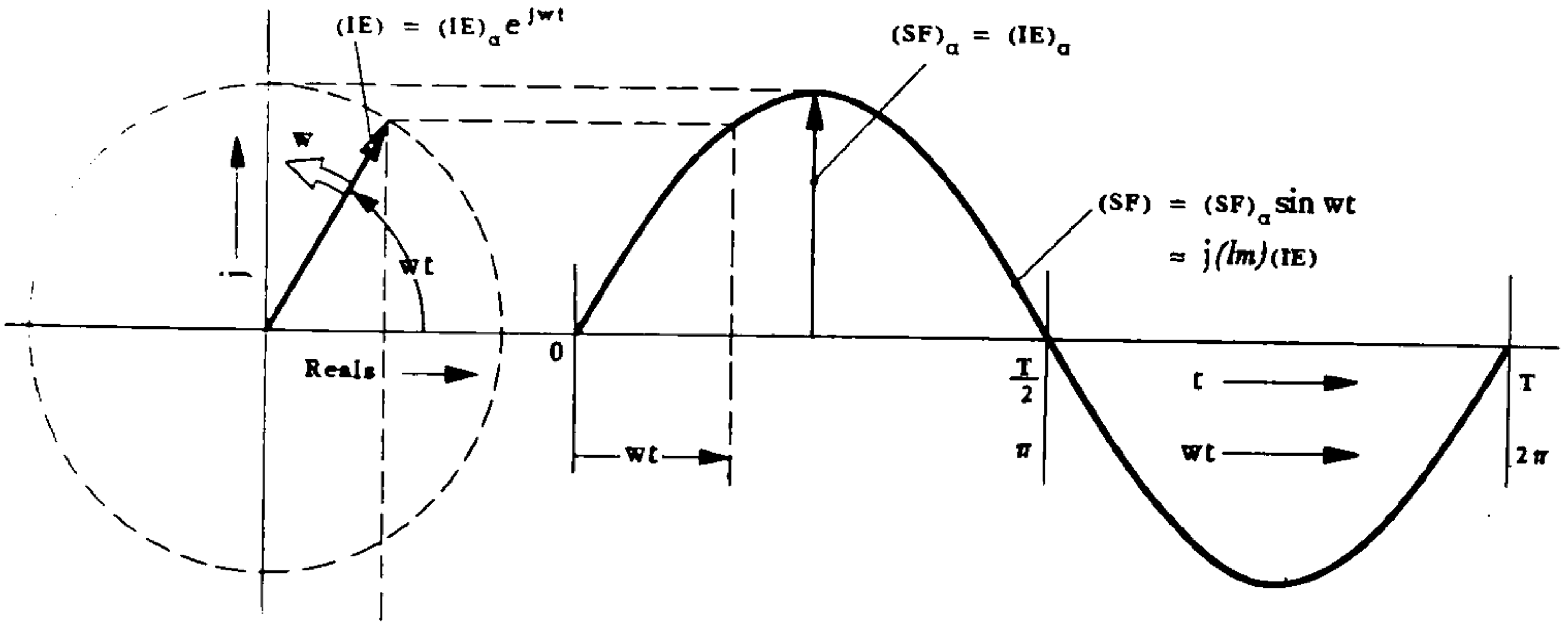
Fig. 12-2. Graphical representation of multiplication and division of imaginary exponentials.

MULTIPLICATION AND DIVISION OF IMAGINARY EXPONENTIALS

12.14. Figure 12-2a illustrates graphically the meaning of the product of two imaginary exponentials. The amplitude of the product is equal to the product of the amplitudes of the factors, and the argument of the product is equal to the sum of the arguments of the factors. Figure 12-2b shows that the division of two exponentials gives a quotient that has its amplitude equal to the ratio of the amplitude of the dividend to the amplitude of the divisor and its argument equal to the difference between the argument of the dividend and the argument of the divisor.

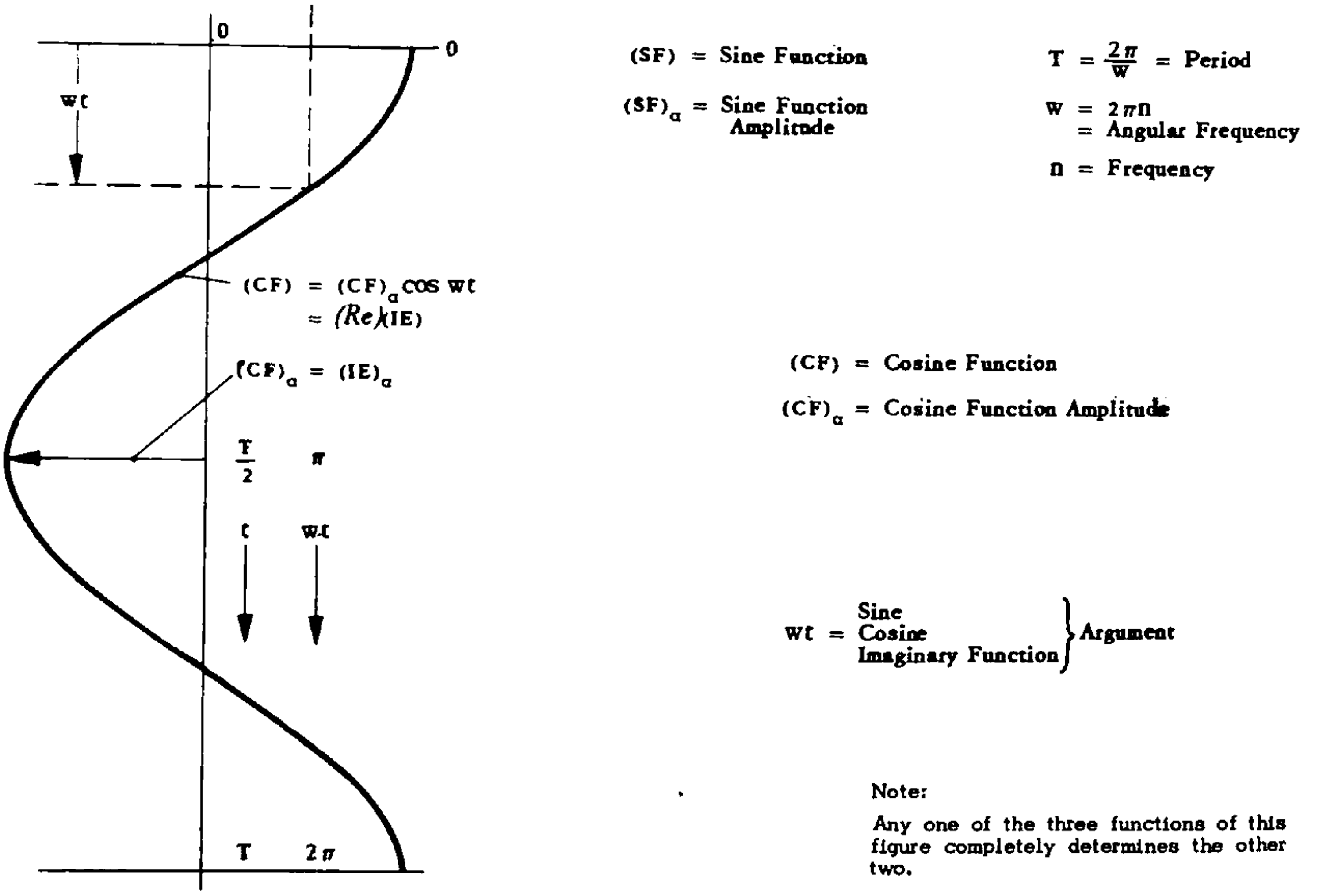
ROTATING VECTOR AND SINUSOIDAL CURVE REPRESENTATIONS OF IMAGINARY EXPONENTIALS WITH TIME-VARYING EXPONENTS

12.15. From the standpoint of practical applications, many important properties of the imaginary exponential appear when the argument is no longer constant but changes with time. As shown in Fig. 12-3, this means that the angle between



a) Rotating vector complex plane representation

b) Sinusoidal function representation



Note:
Any one of the three functions of this figure completely determines the other two.

c) Cosine function representation

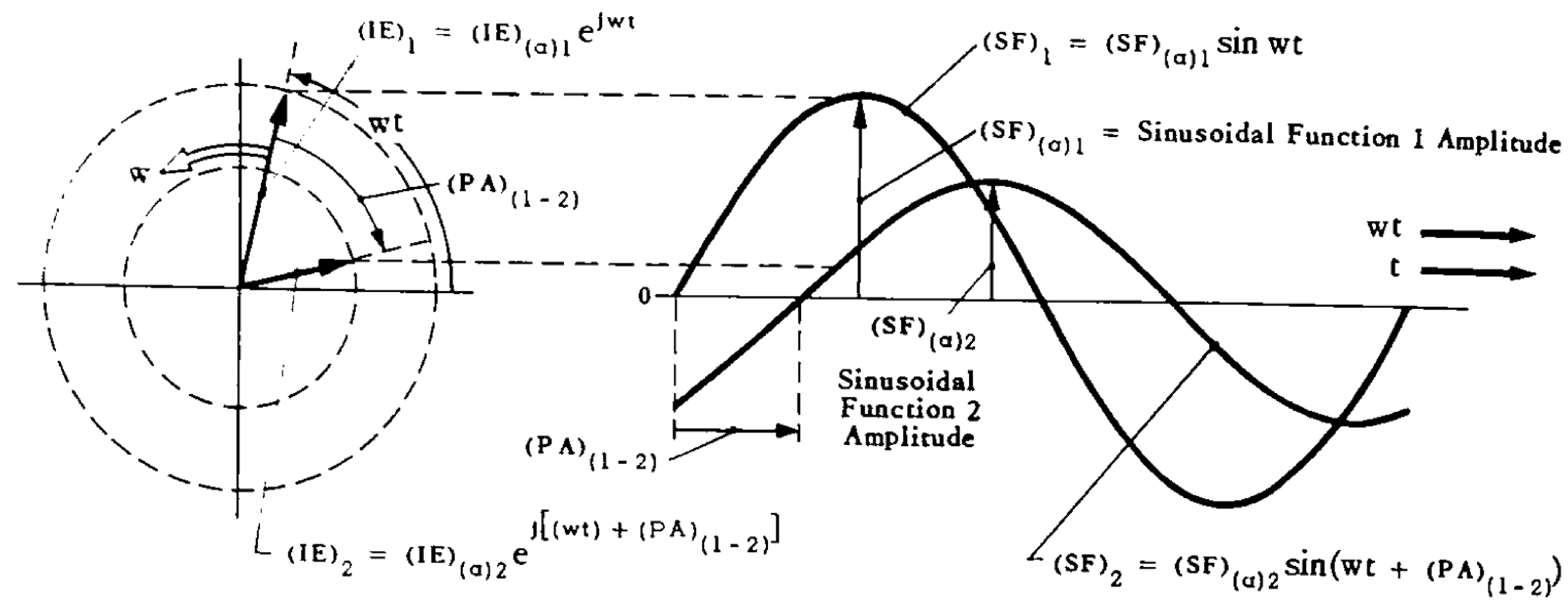
Fig. 12-3. Graphical representation of relationships among imaginary exponentials and corresponding sine and cosine functions having arguments that vary with time.

the positive direction of the axis of reals and the vector representing the imaginary exponential increases linearly with time so that the vector rotates in a counterclockwise direction with a constant angular velocity that has a magnitude equal to w. The real and imaginary components of the exponential are respectively represented in Fig. 12-3 by a cosine curve drawn with the axis of imaginaries as its zero axis, and a sine curve drawn with the real axis as its zero axis. Each of the trigonometric curves completes one cycle in a period equal to the time interval required for the exponential vector to move through one revolution. The angle wt is taken as equal to zero and integral multiples of 2π when the vector is coincident with the axis of reals. Angles and time increase toward the right for the sine curve and downward for the cosine curve.

12.16. The sine curve, the cosine curve, and the rotating vector that represents the corresponding imaginary exponential are always connected by the relationships of Fig. 12-3. This means that the most suitable of the three forms may be applied in finding solutions for any given problem. It is unnecessary to use rigorously complete mathematical expressions to show relationships among the exponential and trigonometric forms as the details of a solution are being worked out. For example, the exponential form is usually best suited for finding the solutions of any differential equation for sinusoidal forcing functions. On the other hand, sine and cosine functions are well adapted for describing and plotting physical quantities as they change with time. In practice, the most convenient form for any given situation is used, with shifts between exponentials and trigonometric functions being written down without including formal details whenever transformations of this type are desirable.

REPRESENTATION OF SINUSOIDAL FUNCTIONS WITH THE SAME FREQUENCY

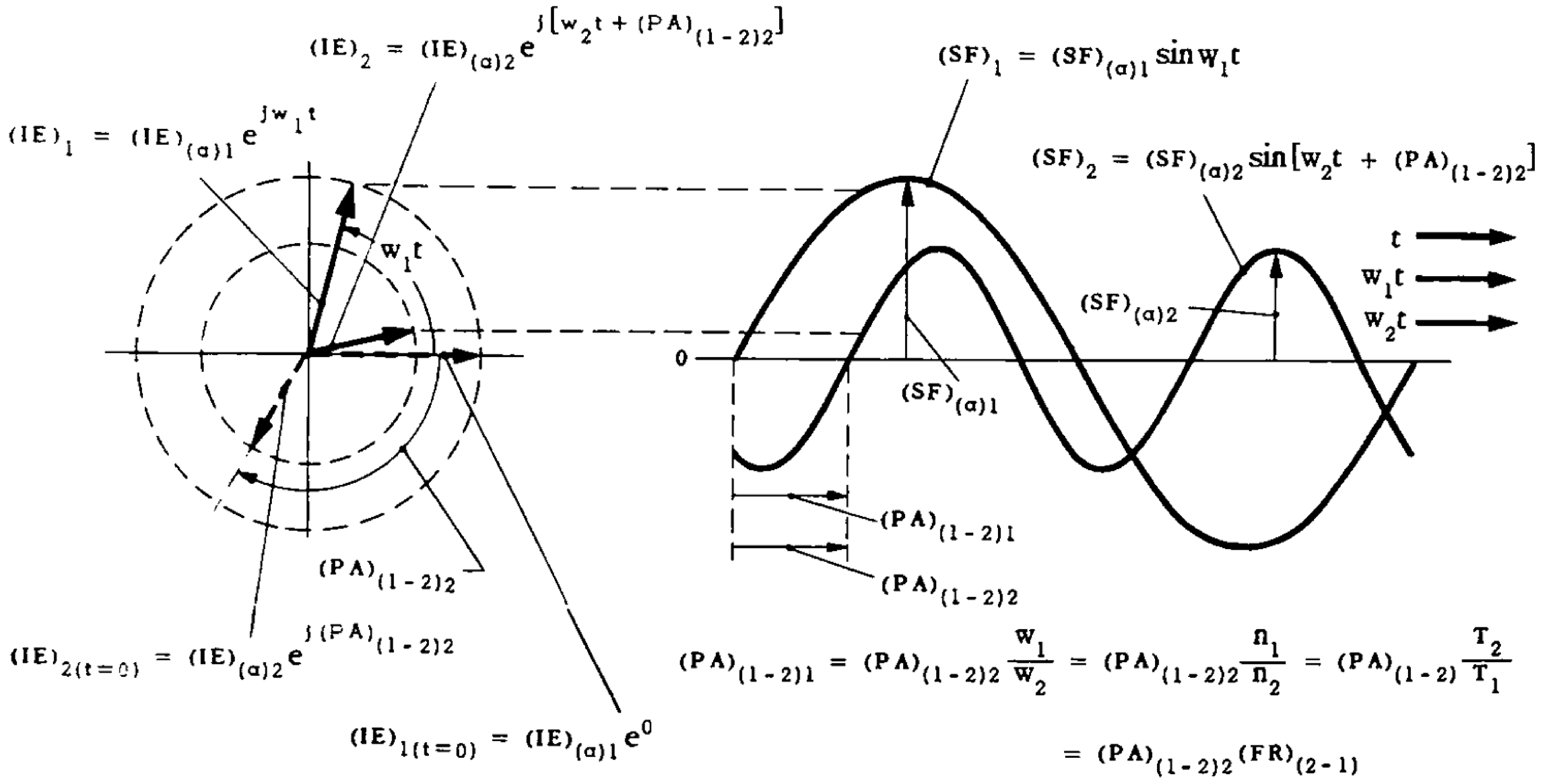
12.17. Figure 12-4a illustrates a very common situation in which the relationship between two sinusoidal variations of the same frequency is important. In this example, the sinusoidal function $(SF)_1$ is taken as the reference function and a second sinusoidal function $(SF)_2$ is the compared function. The rotating vector complex plane diagram corresponding to the sinusoidal variations is shown at the left of the sinusoidal curves. The angular difference between $(IE)_1$, which represents $(SF)_1$, and $(IE)_2$, which represents $(SF)_2$, is the sinusoid 1 – sinusoid 2 phase angle, with the symbol $(PA)_{(1-2)}$ (PA for phase angle, and 1 in the reference subscript position with 2 in the compared quantity position). By the complex plane angle convention, the positive sense for the phase angle is the same as that for the angular velocity, i.e., counterclockwise on the diagram. This means that the phase angle is actually negative as shown in Fig. 12-4. In writing equations that involve phase angles, the symbol (PA) will be written after a plus sign with the understanding that the angle will carry its own sign as required to represent a given function.


$(IE)_1 = (IE)_{(a)1}e^{jwt}$
wt
w
$(PA)_{(1-2)}$
0
$(SF)_1 = (SF)_{(a)1}\sin wt$
$(SF)_{(a)1}$ = Sinusoidal Function 1 Amplitude
wt
t
$(SF)_{(a)2}$
Sinusoidal Function 2 Amplitude
$(PA)_{(1-2)}$
$(IE)_2 = (IE)_{(a)2}e^{j[(wt) + (PA)_{(1-2)}]}$
$(SF)_2 = (SF)_{(a)2}\sin(wt + (PA)_{(1-2)})$

$$(PA)_{(1-2)} = \left\{\begin{matrix}\text{Sinusoid}\\ \text{Exponential}\end{matrix}\right\}1 - \left\{\begin{matrix}\text{Sinusoid}\\ \text{Exponential}\end{matrix}\right\}2 \text{ Phase Angle}$$

The positive sense of the phase angle is counterclockwise on the diagram; for example, the phase angle shown is negative, i.e., a lag angle.

a) Graphical representation of two sinusoids having the same frequency


$(IE)_2 = (IE)_{(a)2}e^{j[w_2t + (PA)_{(1-2)2}]}$
$(IE)_1 = (IE)_{(a)1}e^{jw_1t}$
w_1t
$(PA)_{(1-2)2}$
$(IE)_{2(t=0)} = (IE)_{(a)2}e^{j(PA)_{(1-2)2}}$
$(IE)_{1(t=0)} = (IE)_{(a)1}e^{0}$
$(SF)_1 = (SF)_{(a)1}\sin w_1t$
$(SF)_2 = (SF)_{(a)2}\sin[w_2t + (PA)_{(1-2)2}]$
t
w_1t
w_2t
$(SF)_{(a)2}$
0
$(SF)_{(a)1}$
$(PA)_{(1-2)1}$
$(PA)_{(1-2)2}$

$$(PA)_{(1-2)1} = (PA)_{(1-2)2}\frac{w_1}{w_2} = (PA)_{(1-2)2}\frac{n_1}{n_2} = (PA)_{(1-2)}\frac{T_2}{T_1}$$

$$= (PA)_{(1-2)2}(FR)_{(2-1)}$$

$$(PA)_{(1-2)1} = \left\{\begin{matrix}\text{Sinusoid}\\ \text{Exponential}\end{matrix}\right\}1 - \left\{\begin{matrix}\text{Sinusoid}\\ \text{Exponential}\end{matrix}\right\}2 \text{ Phase Angle}$$

Referred to Exponential 1 Angular Frequency

$$(PA)_{(1-2)2} = \left\{\begin{matrix}\text{Sinusoid}\\ \text{Exponential}\end{matrix}\right\}1 - \left\{\begin{matrix}\text{Sinusoid}\\ \text{Exponential}\end{matrix}\right\}2 \text{ Phase Angle}$$

Referred to Exponential 2 Angular Frequency

$$(FR)_{(1-2)} = \left\{\begin{matrix}\text{Sinusoid}\\ \text{Exponential}\end{matrix}\right\}1 - \left\{\begin{matrix}\text{Sinusoid}\\ \text{Exponential}\end{matrix}\right\}2 \text{ Frequency Ratio}$$

Note:

The term "imaginary" is omitted from the imaginary exponential term because only exponentials of this kind are considered here.

b) Graphical representation of two sinusoids having different frequencies

Fig. 12-4. Representation of imaginary exponentials and the corresponding sinusoidal functions having arguments that vary with time.

REPRESENTATION OF SINUSOIDAL FUNCTIONS WITH DIFFERENT FREQUENCIES

12.18. Situations are common in which the relationship between two sine functions of different amplitudes and frequencies must be described. The method of Fig. 12-4a may be applied here except for the specific difference that the angular frequency w_2 of the second component is not equal to the angular frequency w_1 of the first component, and the phase angle $(PA)_{(1-2)2}$ requires the third subscript 2 to show that the angle is measured in terms of the w_2t angle scale. In order to reduce this phase angle to the scale of the first sinusoidal component, it is necessary to multiply $(PA)_{(1-2)2}$ by the sinusoid 1 – sinusoid 2 frequency ratio, which is by definition equal to the ratio of the frequency of component 1 to the frequency of component 2. The symbol for this ratio is $(FR)_{(2-1)}$, where (FR) stands for frequency ratio; the reference position subscript 2 shows that the reference frequency is that of sinusoid 2; and the compared frequency is that of sinusoid 1. The symbol for the phase angle referred to the component 1 angle scale is $(PA)_{(1-2)1}$. It is important to note that in situations like that illustrated in Fig. 12-4b, the phase angle has a simple meaning on the rotating vector diagram only at instants when the reference vector (vector 1 in Fig. 12-4b) is coincident with the positive real axis. The relationships at an instant of this kind are illustrated by dotted-line vectors in Fig. 12-4b.

FOURIER SERIES

12.19. The Fourier series(7),(10) for representing functions of time is made up of an infinite series of negative imaginary exponential terms, and an infinite series of positive imaginary exponential terms corresponding to integer values of a parameter r that defines the order of each term. The zero value of r corresponds to a constant term. The formal expression for the Fourier series in exponential form is given in section a) of Definition Summary 12-4. Each of the exponential terms in this series has an amplitude and a phase angle, both of which must be properly chosen for each term when the series represents any given function q(t). The angular frequency of the first-order term, which is also called the fundamental, determines the basic frequency for the series. All the other exponential terms have frequencies that are integral multiples of the fundamental frequency. Each of these higher-order terms carries a phase angle with respect to the fundamental* that is expressed in terms of the higher-order angle scale. These conditions are expressed for the r^{th} order term by attaching the subscript (1-r) to the phase angle symbol (PA).

---

* This phase angle may be referred to the angle scale of any given component by changing the comparison quantity subscript to the order of the reference term.

In terms of imaginary exponentials the Fourier series has the form

$$q(t) = \sum_{r=-\infty}^{r=\infty} q_{(a)(e)r} e^{j[rw_1 t + (PA)_{(1-r)r}]} \tag{1}$$

where r has all integer values from $-\infty$ to $+\infty$

$w_1 = \frac{2\pi}{T_1} = 2\pi n_1$ = Angular frequency of first-order component

$q_{(a)(e)r}$ = Amplitude of r^{th} order exponential component

$q(t)$ = Function of time

$(PA)_{(e)(1-r)r}$ = Phase angle of r^{th}-order exponential component with respect to the first-order component measured in r^{th}-order angular units (see Figs. 12-3 and 12-4 for definitions of other symbols)

and

$$q_{(a)(e)r} e^{j(PA)_{(e)(1-r)r}} = \int_0^1 q(t) e^{-j2\pi r(t/T_1)} d(t/T_1) \tag{2}$$

$$q_{(fs)o} = q_{(fs)(cnst)} = \int_0^1 q(t)\, d(t/T) = \text{Fourier series constant (this term is the average value of } q(t)) \tag{3}$$

Dividing through by $q_{(fs)(cnst)}$ and writing the exponential terms as a summation gives

$$\frac{q(t)}{q_{(fs)(cnst)}} = \sum_{r=-\infty}^{r=\infty} (AR)_{(e)(0-r)} e^{j[rw_1 t + (PA)_{(e)(1-r)r}]} \tag{4}$$

where

$$(AR)_{(e)(0-r)} e^{j(PA)_{(e)(1-r)r}} = \frac{\int_0^1 q(t) e^{-j2\pi r(t/T_1)} d(t/T_1)}{\int_0^1 q(t)\, d(t/T_1)} \tag{5}$$

$(AR)_{(e)(0-r)}$ = zero $- r^{th}$ term exponential amplitude ratio

a) Fourier series in terms of imaginary exponentials

With sine and cosine terms the Fourier series becomes

$$q(t) = q_{(fs)(cnst)} + q_{(a)(s)1} \sin(w_1 t) + q_{(a)(s)2} \sin(2w_1 t) + \dots$$
$$+ q_{(a)(c)1} \cos(w_1 t) + q_{(a)(c)2} \cos(2w_1 t) + \dots \tag{6}$$

where

$q_{(a)(s)r}$ = Amplitude of r^{th} sine component

$q_{(a)(c)r}$ = Amplitude of r^{th} cosine component

Definition Summary 12-4. Fourier series forms. (Page 1 of 3)

$$q_{(a)(s)r} = 2\int_0^1 q(t)\sin(2\pi r(t/T_1))\,d(t/T_1) \tag{7}$$

$$q_{(a)(c)r} = 2\int_0^1 q(t)\cos(2\pi r(t/T_1))\,d(t/T_1) \tag{8}$$

$$q_{(fs)(cnst)} = \int_0^1 q(t)\,d(t/T_1) \tag{9}$$

Dividing through by $q_{(fs)(cnst)}$ and summing sine and cosine series gives

$$\begin{aligned}\frac{q(t)}{q_{(fs)(cnst)}} &= 1 + \sum_{r=1}^{r=\infty} (AR)_{(s)(0-r)}\sin(r w_1 t) \\ &\quad + \sum_{r=1}^{r=\infty} (AR)_{(c)(0-r)}\cos(r w_1 t)\end{aligned} \tag{10}$$

$$(AR)_{(s)(0-r)} = \frac{q_{(a)(s)r}}{q_{(fs)(cnst)}} = \text{Zero-to-r}^{\text{th}}\text{ sine term amplitude ratio} \tag{11}$$

$$(AR)_{(c)(0-r)} = \frac{q_{(a)(c)r}}{q_{(fs)(cnst)}} = \text{Zero-to-r}^{\text{th}}\text{ cosine term amplitude ratio} \tag{12}$$

b) Fourier series in the sine-and-cosine term form

Expressed in sine terms and phase angles, the Fourier series becomes

$$q(t) = q_{(fs)(cnst)} + q_{(a)(s)(pa)1}\sin[w_1 t] + \sum_{r=1}^{r=\infty} q_{(a)(s)(pa)r}\sin[r w_1 t + (PA)_{(s)(1-r)r}] \tag{13}$$

where

$$q_{(a)(s)(pa)r} = \sqrt{q^2_{(a)(c)r} + q^2_{(a)(s)r}} = \text{r}^{\text{th}}\text{ sine}-\text{phase angle term amplitude} \tag{14}$$

$$(PA)_{(s)(1-r)r} = \tan^{-1}\frac{q_{(a)(c)r}}{q_{(a)(s)r}} = \text{First-to-r}^{\text{th}}\text{ sine term phase angle expressed in r}^{\text{th}}\text{ term angle units} \tag{15}$$

Dividing through by $q_{(fs)(cnst)}$ and summing the sine terms gives

$$\frac{q(t)}{q_{(fs)(cnst)}} = 1 + \sum_{r=1}^{r=\infty} (AR)_{(s)(pa)(0-r)}\sin[r w_1 t + (PA)_{(s)(1-r)r}] \tag{16}$$

$$(AR)_{(s)(pa)(0-r)} = \frac{q_{(a)(s)(pa)r}}{q_{(fs)(cnst)}} = \text{Zero-to-r}^{\text{th}}\text{ sine}-\text{phase angle term amplitude ratio} \tag{17}$$

c) Fourier series in the sine – phase angle term form

Definition Summary 12-4. Fourier series forms. (Page 2 of 3)

Expressed in cosine terms and phase angles, the Fourier series becomes

$$q(t) = q_{(fs)(cnst)} + \sum_{r=1}^{r=\infty} q_{(a)(c)(pa)r} \cos[r w_1 t + (PA)_{(c)(1-r)r}]^{*} \quad (18)$$

where

$$q_{(a)(c)(pa)r} = \sqrt{q^2_{(a)(c)r} = q^2_{(a)(s)r}} = r^{th} \text{ cosine -- phase angle form amplitude ratio} \quad (19)$$

$$(PA)_{(c)(1-r)r} = \tan^{-1} \frac{-q_{(a)(s)r}}{q_{(a)(c)r}} = \text{First-to-}r^{th} \text{ cosine term phase angle in } r^{th} \text{ term angle units}^{*} \quad (20)$$

Dividing through by $q_{(fs)(cnst)}$ and summing the cosine terms gives

$$\frac{q(t)}{q_{(fs)(cnst)}} = 1 + \sum_{r=1}^{r=\infty} (AR)_{(c)(pa)(0-r)} \cos[r w_1 t + (PA)_{(c)(1-r)r}] \quad (21)$$

where

$$(AR)_{(c)(pa)(0-r)} = \frac{q_{(a)(c)(pa)r}}{q_{(fs)(cnst)}} = \text{Zero-to-}r^{th} \text{ cosine -- phase angle term amplitude ratio} \quad (22)$$

---

* This differs from usage by others (for example, see Wylie (37), pages 129 to 132) who write the series as

$$q(t) = q_{(fs)(cnst)} + \sum_{r=1}^{r=\infty} q_{(a)(c)(pa)r} \cos[r w_1 t - (PA)_{(c)(1-r)r}]$$

and define

$$(PA)_{(c)(1-r)r} = \tan^{-1} \frac{q_{(a)(s)r}}{q_{(a)(c)r}}$$

d) Fourier series in the cosine – phase angle term form

Definition Summary 12-4. Fourier series forms. (Page 3 of 3)

IMAGINARY EXPONENTIAL TERM FORM OF FOURIER SERIES

12.20. It is shown in texts* on Fourier series that the imaginary exponential corresponding to the product of the r^{th} order component amplitude and the exponential with $j(PA)_{(e)(1-r)r}$ as the exponent is equal to the integral from 0 to unity with respect to the ratio (t/T_1) of q(t), multiplied by e raised to the $-j2\pi r(t/T_1)$ power. The limits of integration correspond to the beginning and end of one period T_1 of the fundamental. The constant term of the Fourier series is found by an integral similar to those for the exponential terms except that the exponential is omitted from the integrand.

ORTHOGONALITY PROPERTIES OF IMAGINARY EXPONENTIALS AND TRIGONOMETRIC FUNCTIONS

12.21. The mathematical relationships** that establish the integrals to determine amplitudes and phase angles for the exponential terms that may be used to represent any given function q(t) are summarized in Definition Summary 12-5.

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* See references (7), (10).

** These relationships are rigorously derived in the previously cited books by Byerley (7) and Churchill (10).

For exponential terms, the essential fact is that if the product of two exponentials with exponents that differ only in multiplying integers is integrated between zero and unity with respect to the ratio (t/T), the result is zero if the integers are different, and unity if the integers are alike in magnitude but have different signs. This fact may be verified by direct substitution in Eq. (1) of Definition Summary 12-5.

12.22. The equations of Definition Summary 12-5 are all written with the ratio t/T (where T is in general a "reference period") as the variable of integration. This choice of the integration variable is made in order to place all operations in nondimensional terms and to give the simple integration limits of zero and unity for the time interval corresponding to one reference period.

12.23. Equations (2), (3), and (4) of Definition Summary 12-5 show that the typical behavior described above for integrals involving imaginary exponentials also holds for integrals with sines or cosines replacing the exponentials. For example, the integral over one complete period with respect to t/T of the product of a sine and a cosine with integer coefficients multiplying $2\pi t/T$ in the arguments of the trigonometric terms is always zero. When the integrand is a product of sines, or a product of cosines, the same result holds except when the arguments for the two terms of the products contain the same integer. In this case, the value of the integral over one complete period is equal to 1/2.

12.24. Functions that show properties like those discussed above for imaginary exponentials, sines, and cosines are called orthogonal functions. The requirements that apply to all orthogonal functions are given in general terms by section b) of Definition Summary 12-5. In addition to the exponential and trigonometric forms, many functions have the property of orthogonality and may be used for describing physical quantities. Many of these functions are discussed in treatises on mathematical analysis.(36) Because the mathematics required to process exponential and trigonometric functions is particularly simple and is easily generalized to describe situations with all degrees of complexity, forms of this type are very widely used for describing variations of physical quantities. The almost universal applicability of exponentials, sines, and cosines to the problems of instrument engineering dictates the decision to give these functions a position of primary importance in this book.

NONDIMENSIONAL FORM OF FOURIER SERIES WITH IMAGINARY EXPONENTIAL TERMS

12.25. When both sides of the exponential Fourier series of Eq. (1) of Definition Summary 12-4 are divided by the constant term, and the separate exponential terms are brought together in a summation, the form of Eq. (4) of the definition summary appears. Each exponential term is multiplied by the zero – r^{th}

The coefficient-determining equations of Definition Summary 12-4 depend upon the following integral relationships:*

$$\int_0^1 e^{j2\pi r(t/T)} e^{j2\pi s(t/T)} d(t/T) = 0 \quad \text{when } r \neq -s \qquad (1)$$
$$= 1 \quad \text{when } r = -s$$
$$r,\ s \text{ are positive integers}$$

$$\int_0^1 \cos(2\pi r(t/T)) \sin(2\pi s(t/T)) d(t/T) = 0 \text{ for } r,\ s \text{ any value} \qquad (2)$$

$$\int_0^1 \cos(2\pi r(t/T)) \cos(2\pi s(t/T)) d(t/T) = \tfrac{1}{2} \text{ when } r = s \qquad (3)$$
$$= 0 \text{ when } r \neq s$$

$$\int_0^1 \sin(2\pi r(t/T)) \sin(2\pi s(t/T)) d(t/T) = \tfrac{1}{2} \text{ when } r = s \qquad (4)$$
$$= 0 \text{ when } r \neq s$$

The above results are unchanged if $\begin{cases} \cos(2\pi r(t/T)) \text{ becomes } \cos(2\pi r(t/T) \pm A) \\ \sin(2\pi r(t/T)) \text{ becomes } \sin(2\pi r(t/T) \pm A) \end{cases}$
where A is a constant angle.

a) Orthogonal relationships for exponential and trigonometric functions

By definition, the set of functions

$$f_1(t/T);\ f_2(t/T);\ \ldots\ f_n(t/T)$$

forms an orthogonal function system if

1) all functions are continuous in the interval $a < t/T < b$

2) $$\int_a^b |f_r(t/T)|\, d(t/T), \text{ and} \int_a^b |f_r(t/T)|^2 d(t/T), \text{ are finite} \qquad (5)$$

3) $$\int_a^b f_r(t/T) f_s(t/T)\, d(t/T) > 0 \text{ when } r = s \qquad (6)$$
$$= 0 \text{ when } r \neq s$$

In general, $q(t/T)$ may be expressed in terms of an orthogonal function system if it is possible to write

$$q(t/T) = \sum_{s=1}^{s=\infty} q_s f_s(t/T) \qquad (7)$$

b) Definition of orthogonal function system

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* See A Short Table of Integrals, Peirce (30).

Definition Summary 12-5. Generalized orthogonal relationships.

order amplitude ratio, with the main symbol (AR) and a subscript with zero (to represent the zeroth -order term) in the reference position, and r (to represent the rth order) in the compared quantity position. The amplitude ratio subscript convention makes it possible to represent the amplitude ratio resulting from the use of the amplitude of any term as the reference quantity in forming the ratio. The defining relationship for the amplitude ratio and phase angle of a typical Fourier series term is given as Eq. (5) of Definition Summary 12-4.

SINE AND COSINE TERM FORMS OF FOURIER SERIES

12.26. The Fourier series may be written in the form of Eq. (6) of Definition Summary 12-4, which has a constant term, an infinite series of sine terms, and an infinite series of cosine terms. In this sine-and-cosine Fourier series, no phase angles appear in the arguments of the trigonometric terms. Equations (7), (8) and (9) of Definition Summary 12-4 give the integrals that relate the amplitudes of individual terms to the function q(t) and the order of the term considered. When each term of the sine-and-cosine Fourier series is divided by the constant term, the result is the nondimensional series of Eq. (10) of Definition Summary 12-4. Equations (7), (8) and (9) of Definition Summary 12-4 summarize the integral relationships that determine the amplitudes for individual terms. The nondimensional form of the sine-and-cosine Fourier series appears when each term is divided by the constant term of the series, and separate summations are written for the sine terms and the cosine terms. The defining relation for the zero – r^{th} sine term amplitude ratio is given as Eq. (11) of Definition Summary 12-4. Equation (12) of the definition summary is the corresponding form for the zero – r^{th} cosine term amplitude ratio.

12.27. Equation (13) of Definition Summary 12-4 gives the sine – phase angle form of the Fourier series. This form consists of a constant term and an infinite series of sine terms. Except for the first-order term, the arguments of all the sine terms include a first – r^{th} sine term phase angle in the r^{th} component angle scale. The symbols for these phase angles are similar to the corresponding phase angle symbols as defined for the exponential series. The difference between the sine term phase angle symbols and the exponential phase angle symbols is that the (e) subscript (for exponential) is replaced by (s) (for sine). Equation (14) of Definition Summary 12-4 gives the sine – phase angle series amplitudes in terms of the sine – cosine series amplitudes. Equation (15) of this definition summary shows the relation between the sine – phase angle series phase angles and the sine and cosine term amplitudes. The nondimensional summation form of the sine – phase angle Fourier series appears when each term is divided by the constant term. This form is shown by Eq. (16) of Definition Summary 12-4, and the defining relationship for the amplitude ratio is given in Eq. (17).

12.28. The cosine – phase angle form of the Fourier series is shown by Eqs. (18) and (21) of Definition Summary 12-4, and the amplitude, phase angle and amplitude ratio definitions are summarized in Eqs. (19), (20) and (22). The cosine – phase angle series is identical with the sine – phase angle series except that cosines instead of sines are used in the terms of the summation.

CHARACTERISTICS OF FOURIER SERIES FORMS

12.29. In practice, it is often desirable to transform the representation of a given function among the four types of Fourier series that are listed in Definition

Form of Series	Typical Term Parameters	Exponential Phase Angle Form		Sine-Cosine Form
		Amplitude	Phase Angle	Sine Amplitude
Exponential-Phase Angle	$q_{(a)(e)r}$ $(PA)_{(e)(1-r)r}$	$q_{(a)(e)r}$	$(PA)_{(e)(1-r)r}$ **	$-2q_{(a)r}\sin(PA)_{(e)(1-r)r}$
Sine-Cosine	$q_{(a)(s)r}$ $q_{(a)(c)r}$	$\frac{1}{2}\sqrt{q^2_{(a)(s)r}+q^2_{(a)(c)r}}$ *	$\tan^{-1}\frac{q_{(a)(s)r}}{q_{(a)(c)r}}$ **	$q_{(a)(s)r}$
Sine-Phase Angle	$q_{(a)(s)(PA)r}$ $(PA)_{(s)(1-r)r}$	$\frac{1}{2}q_{(a)(s)(PA)r}$ *	$(PA)_{(s)(1-r)r}-\frac{\pi}{2}$ **	$q_{(a)(s)(PA)r}\cos(PA)_{(s)(1-r)r}$
Cosine-Phase Angle	$q_{(a)(c)(PA)r}$ $(PA)_{(c)(1-r)r}$	$\frac{1}{2}q_{(a)(c)(PA)r}$ *	$(PA)_{(c)(1-r)r}$ **	$-q_{(a)(c)(PA)r}\sin(PA)_{(c)(1-r)r}$

* For r different from zero only, the constant term (r = 0) is the same for all forms of the series.

** For r positive only. For negative r, the sign of (PA) is negative.

*** The phase angle carries its own sign.

Table 12-1

Fourier series amplitude and phase angle conversions.

Summary 12-4. For example, if the function q(t) is available as an experimental curve, and has its Fourier series components determined by an analyzer of the Henrici-Coradi(26) type, the data naturally appear in the form of amplitudes for a sine – cosine series. In order to check the degree of approximation to the actual function q(t) by any given number of terms, it is convenient to make plots showing the resultant of a group of terms either graphically or by means of a Fourier synthesizer.(13),(31) This approximation is most easily found by summing components plotted in the sine – phase angle form or the cosine – phase angle form. With the Fourier series representation for any given function established, the response of any operating system to the given function as an input is most easily found by carrying out mathematical operations on series terms of the imaginary exponential form. Because all of the situations mentioned in this paragraph will often appear in the same problem, facility in using the various Fourier series forms is a considerable advantage to the instrument engineer.

TABULATION OF RELATIONSHIPS AMONG AMPLITUDES AND PHASE ANGLES OF FOURIER SERIES FORMS

12.30. Time is saved and mistakes are avoided in problems that fit into the pattern outlined by the preceding paragraph if a tabulation is available that lists the relationships among the amplitudes and phase angles of corresponding terms in the various Fourier series forms. A summary of this kind is provided by

Sine-Cosine Form	Sine-Phase Angle Form		Cosine-Phase Angle Form	
Cosine Amplitude	Amplitude	Phase Angle	Amplitude	Phase Angle
$2q_{(a)r} \cos (PA)_{(e)(1-r)r}$	$2q_{(a)(e)r}$	$(PA)_{(e)(1-r)r} + \frac{\pi}{2}$	$2q_{(a)(e)r}$	$(PA)_{(e)(1-r)r}$
$q_{(a)(c)r}$	$\sqrt{q^2_{(a)(s)r} + q^2_{(a)(c)r}}$ *	$\tan^{-1} \frac{q_{(a)(c)r}}{q_{(a)(s)r}}$	$\sqrt{q^2_{(a)(s)r} + q^2_{(a)(c)r}}$ *	$\tan^{-1} \frac{q_{(a)(s)r}}{q_{(a)(c)r}}$ ***
$q_{(a)(s)(PA)r} \sin (PA)_{(s)(1-r)r}$	$q_{(a)(s)(PA)r}$	$(PA)_{(s)(1-r)r}$	$q_{(a)(s)(PA)r}$	$(PA)_{(s)(1-r)r} - \frac{\pi}{2}$
$q_{(a)(c)(PA)r} \cos (PA)_{(c)(1-r)r}$	$q_{(a)(c)(PA)r}$	$(PA)_{(c)(1-r)r} + \frac{\pi}{2}$	$q_{(a)(c)(PA)r}$	$(PA)_{(c)(1-r)r}$

Table 12-1 (cont.)

Table 12-1, which gives the equations for the amplitude and phase angle for the r^{th} component of the series form listed at the top of any column in terms of the amplitudes and phase angles for the series form listed in the row at the left of the table. For example, under the exponential – phase angle form, the amplitude of the r^{th} term is shown as equal to one-half the square root of the sum of the squares of the amplitudes of the r^{th} terms of the sine and cosine forms.

FOURIER SERIES REPRESENTATION OF THE SQUARE WAVE AND THE PULSE

12.31. The Fourier series method of representing physical quantities may be illustrated by examples that are also often useful for describing practical situations. The mathematical forms required to describe the series of rectangular shapes that make up a typical square wave* are tabulated in Data Summary 12-1, which also includes diagrams and specifications for the wave itself. The specifying equations are given for each of the four Fourier series forms of Definition Summary 12-4. Figure 12-5 shows the square wave as a heavy, dashed line, the resultant of the first three Fourier series terms as a heavy, full line, and three sinusoidal components as light, dashed lines to illustrate the representation of a function by Fourier series terms. It is interesting to note that the square wave is formed as the sum of all odd-order sinusoidal components, each of which has an amplitude of $4/r\pi$.

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* The term "wave" is applied to shapes that represent variations of some quantity with time because the particles of a fluid subjected to displacements corresponding to the function would move in a way described by the "wave shape."

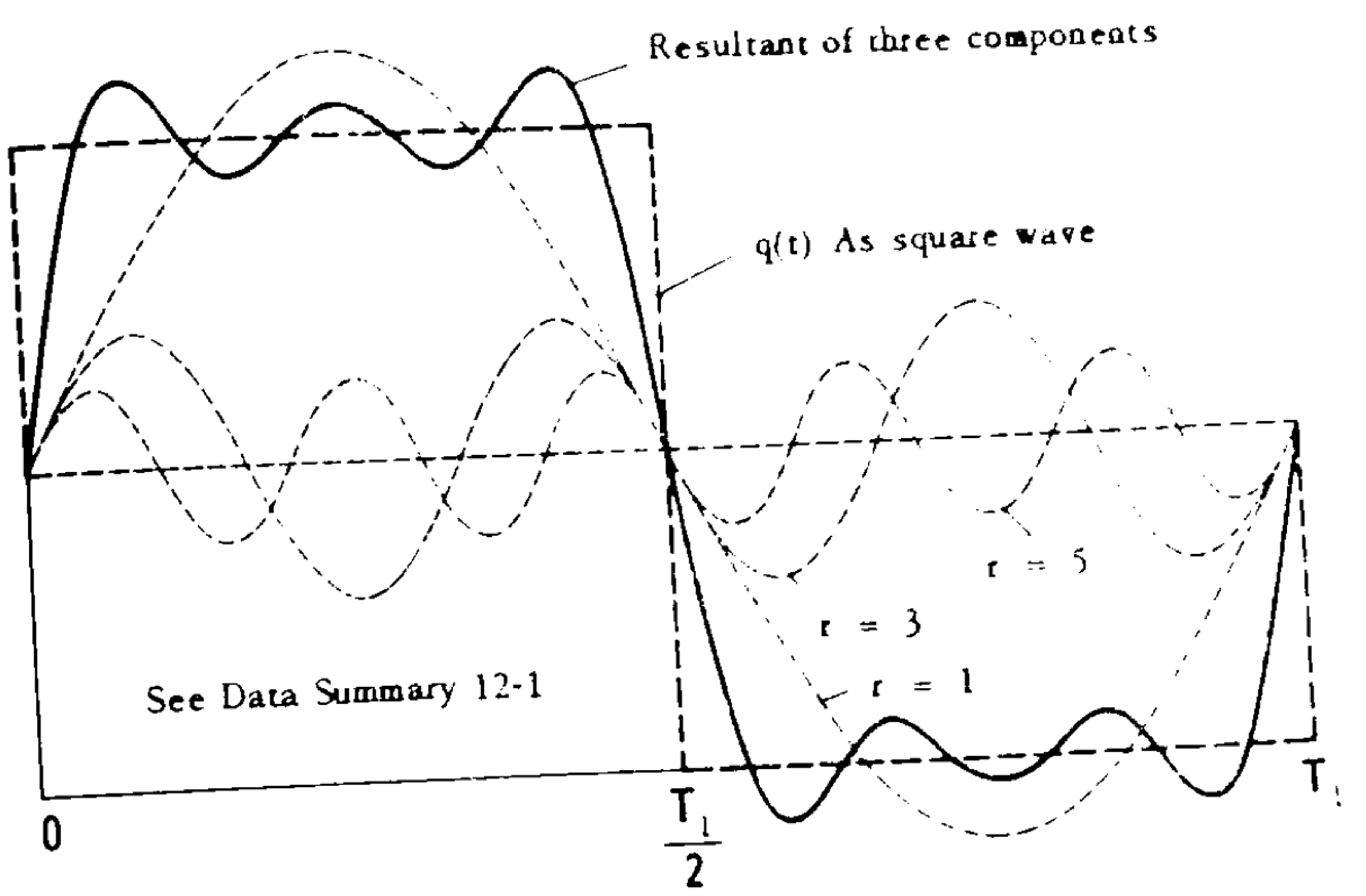
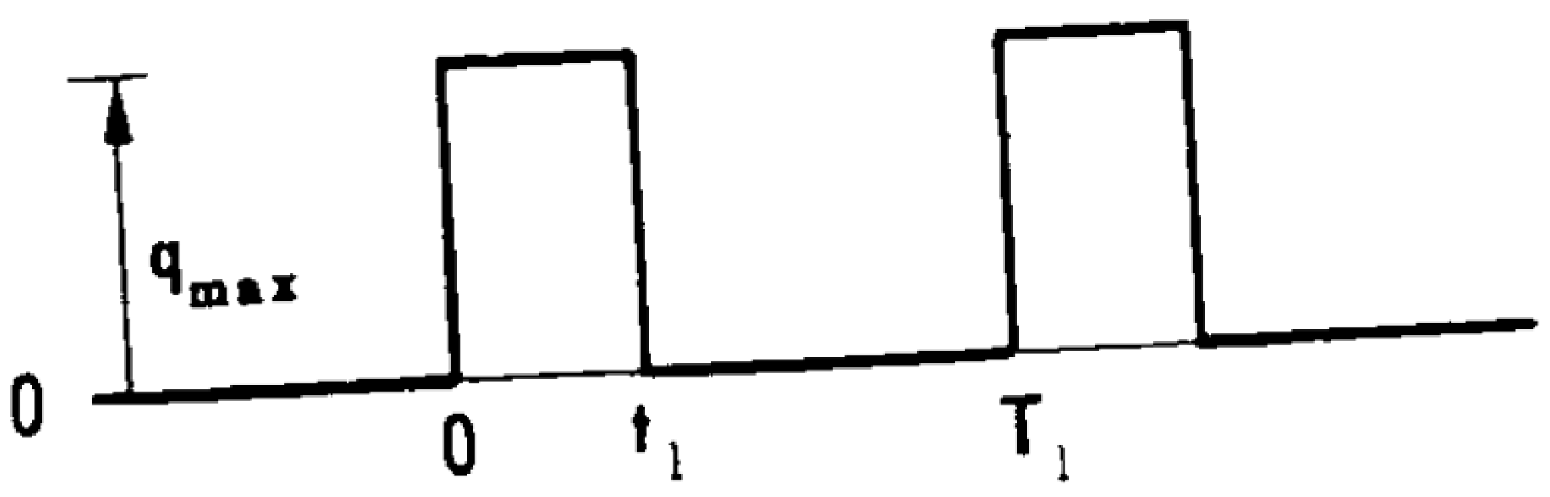


Fig. 12-5. Square wave approximation by first three Fourier series terms.



Definition:

$$q(t) = q_{max} \qquad 0 < t < t_1$$
$$q(t) = 0 \qquad t_1 < t < T_1 \qquad (1)$$
$$q(t) = \frac{q_{max}}{2} \qquad t = 0, t_1, T_1$$

Exponential series representation:

$$q(t) = q_{max} \frac{t_1}{T_1} \sum_{r=-\infty}^{\infty} \frac{\sin r\pi \frac{t_1}{T_1}}{r\pi \frac{t_1}{T_1}} e^{j(rw_1 t - r\pi \frac{t_1}{T_1})} \qquad (2)$$

Sine – cosine series representation:

$$q(t) = q_{max} \frac{t_1}{T_1} [1 + \sum_{r=1}^{\infty} \frac{1 - \cos r2\pi \frac{t_1}{T_1}}{r\pi \frac{t_1}{T_1}} \sin rw_1 t + \sum_{r=1}^{\infty} \frac{\sin r2\pi \frac{t_1}{T_1}}{r\pi \frac{t_1}{T_1}} \cos rw_1 t] \qquad (3)$$

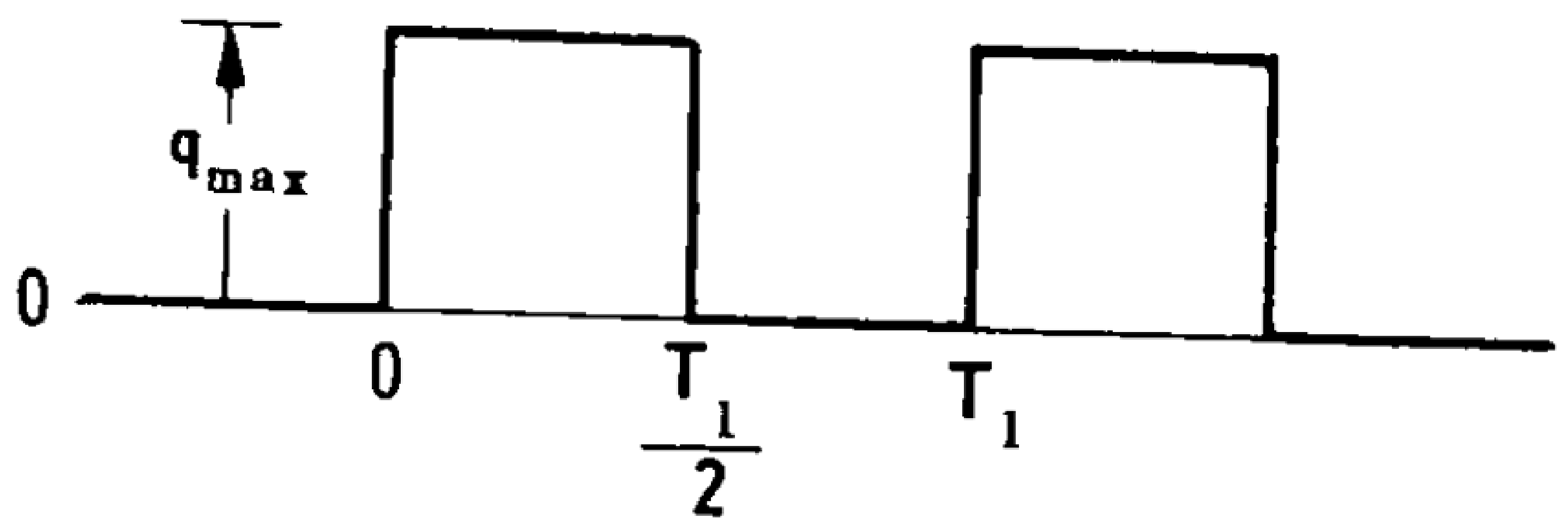
Sine – phase angle series representation:

$$q(t) = q_{max} \frac{t_1}{T_1} [1 + \sum_{r=1}^{\infty} \frac{2 \sin r\pi \frac{t_1}{T_1}}{r\pi \frac{t_1}{T_1}} \sin (rw_1 t + \frac{\pi}{2} - r\pi \frac{t_1}{T_1})] \qquad (4)$$

Cosine – phase angle series representation:

$$q(t) = q_{max} \frac{t_1}{T_1} [1 + \sum_{r=1}^{\infty} \frac{2 \sin r\pi \frac{t_1}{T_1}}{r\pi \frac{t_1}{T_1}} \cos (rw_1 t - r\pi \frac{t_1}{T_1})] \qquad (5)$$

Data Summary 12-2. Fourier series equations for rectangular pulse wave.



Definition:

$$q(t) = q_{max} \quad 0 < t < \frac{T_1}{2}$$

$$q(t) = 0 \quad \frac{T_1}{2} < t < T_1 \qquad (1)$$

$$q(t) = \frac{q_{max}}{2} \quad t = 0, \frac{T_1}{2}, T_1$$

Exponential series representation:

$$q(t) = \frac{q_{max}}{2} \sum_{r=-\infty}^{\infty} \frac{2}{r\pi} e^{j(rw_1 t - \pi/2)} \qquad (2)$$

r = odd integers

Sine – cosine series representation:

$$q(t) = \frac{q_{max}}{2} [1 + \sum_{r=1}^{\infty} \frac{4}{r\pi} \sin (rw_1 t)] \qquad (3)$$

r = positive odd integers

Sine – phase angle series representation:

$$q(t) = \frac{q_{max}}{2} [1 + \sum_{r=1}^{\infty} \frac{4}{r\pi} \sin (rw_1 t)] \qquad (4)$$

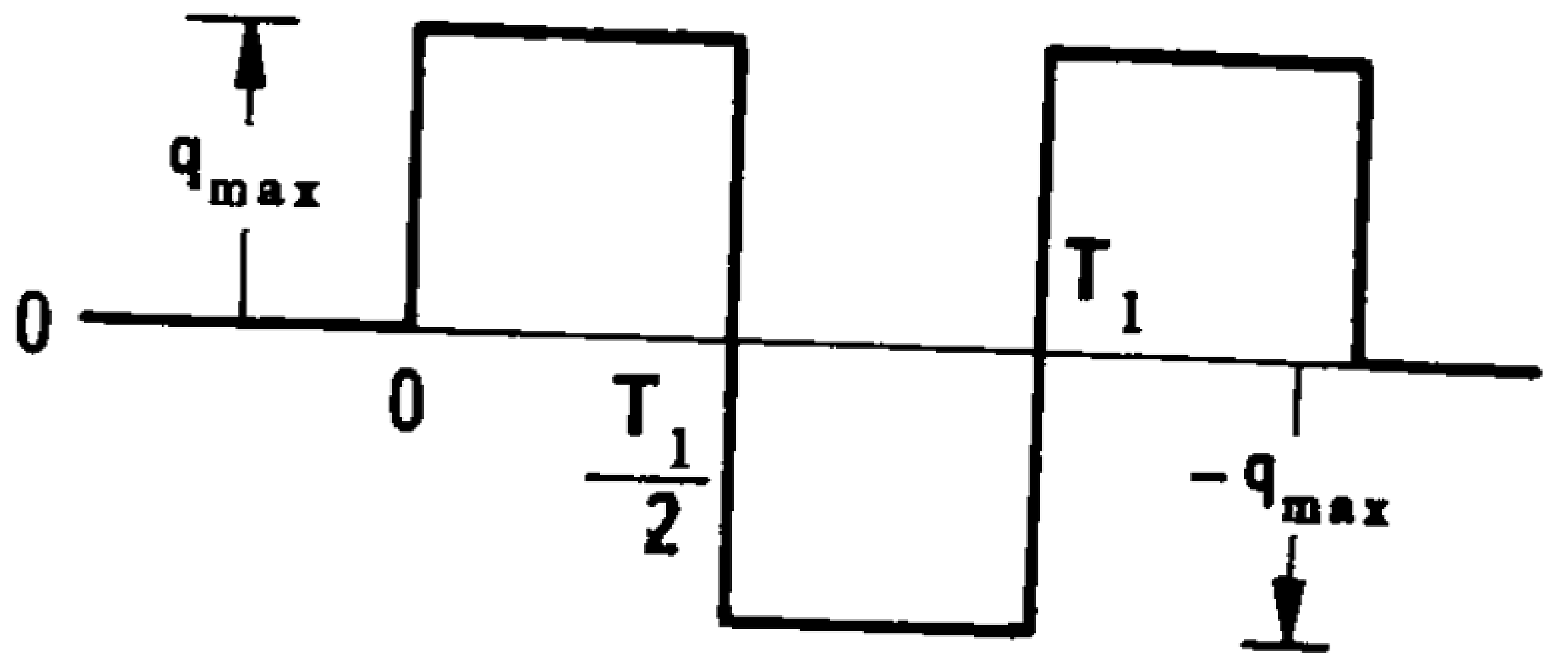
r = positive odd integers

Cosine – phase angle series representation:

$$q(t) = \frac{q_{max}}{2} [1 + \sum_{r=1}^{\infty} \frac{4}{r\pi} \cos (rw_1 t - \frac{\pi}{2})] \qquad (5)$$

r = positive odd integers

a) Fourier series representation for square wave with zero level at bottom of wave



Definition:

$$q(t) = q_{max} \quad 0 < t < \frac{T_1}{2}$$

$$q(t) = -q_{max} \quad \frac{T_1}{2} < t < T_1 \qquad (6)$$

$$q(t) = 0 \quad t = 0, \frac{T_1}{2}, T_1$$

Exponential series representation:

$$q(t) = q_{max} \sum_{r=-\infty}^{\infty} \frac{2}{r\pi} e^{j(rw_1 t - \pi/2)} \qquad (7)$$

r = odd integers

Sine – cosine series representation:

$$q(t) = q_{max} \sum_{r=1}^{\infty} \frac{4}{r\pi} \sin (rw_1 t) \qquad (8)$$

r = positive odd integers

Sine – phase angle series representation:

$$q(t) = q_{max} \sum_{r=1}^{r=\infty} \frac{4}{r\pi} \sin (rw_1 t) \qquad (9)$$

r = positive odd integers

Cosine – phase angle series representation:

$$q(t) = q_{max} \sum_{r=1}^{\infty} \frac{4}{r\pi} \cos (rw_1 t + \frac{3\pi}{2}) \qquad (10)$$

r = positive odd integers

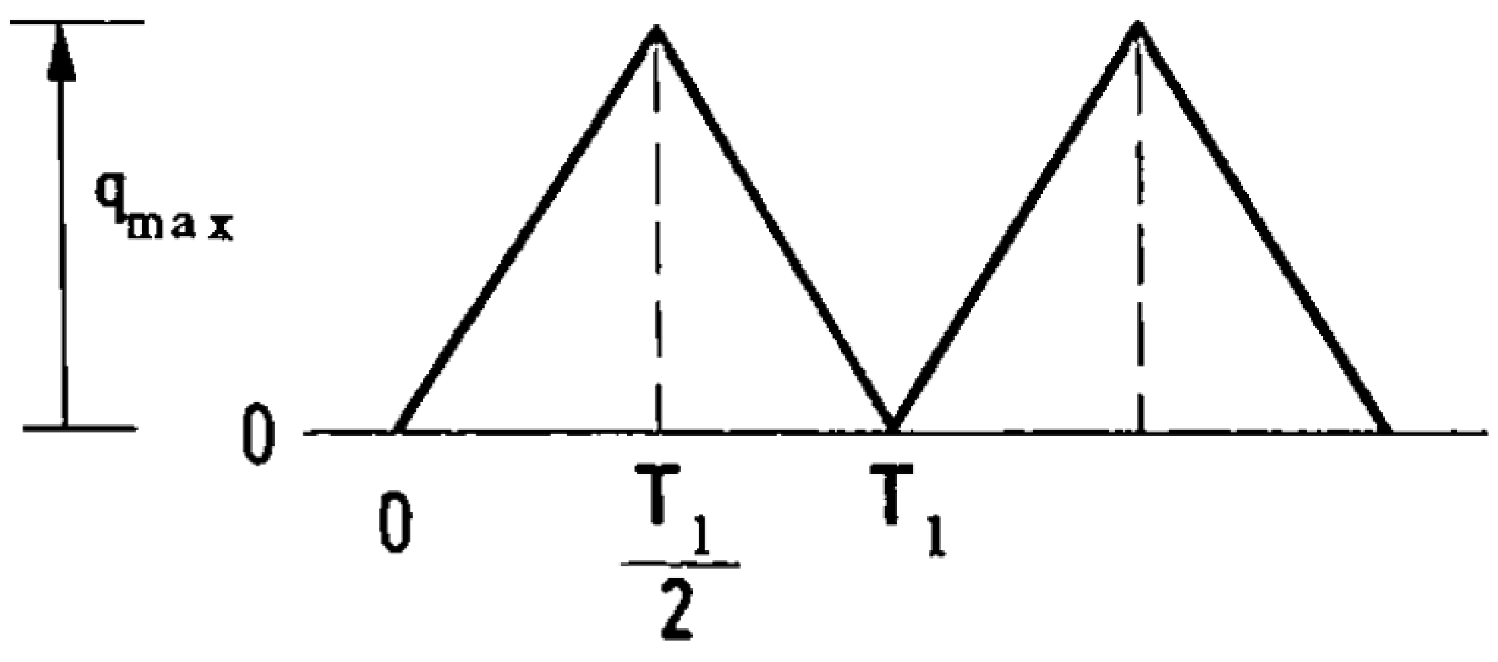
b) Fourier series for square wave symmetrical about zero level

Term		
	1	$\frac{4}{\pi} = 1.2732$
	3	$\frac{4}{3\pi} = 0.4244$
	5	$\frac{4}{5\pi} = 0.255$

$rw_1 t$ (deg)	$\sin w_1 t$	$\frac{4}{\pi} \sin w_1 t$	$\frac{4}{3\pi} \sin 3w_1 t$	$\frac{4}{5\pi} \sin 5w_1 t$
0	0.000	0.000	0.000	0.000
15	0.259	0.330	0.110	0.066
30	0.500	0.638	0.213	0.128
45	0.707	0.903	0.301	0.181
60	0.866	1.105	0.368	0.221
75	0.959	1.223	0.408	0.245
90	1.000	1.273	0.424	0.255

c) Tabulation of numerical data for Fourier series for square wave

Data Summary 12-1. Fourier series equations for square wave.



Definition:

$$q(t) = \frac{2t}{T_1} q_{max}, \qquad 0 < t < \frac{T_1}{2}$$

$$q(t) = 2[1 - \frac{t}{T_1}]q_{max}, \frac{T_1}{2} < t < T_1 \tag{1}$$

Exponential series representation:

$$q(t) = \frac{q_{max}}{2} \sum_{r=-\infty}^{\infty} \frac{4}{(r\pi)^2} e^{j(rw_1t - \pi)} \tag{2}$$

r = odd integers

Sine – cosine series representation:

$$q(t) = \frac{q_{max}}{2} [1 - \sum_{r=1}^{\infty} \frac{8}{(r\pi)^2} \cos rw_1t] \tag{3}$$

r = odd positive integers

Sine – phase angle series representation:

$$q(t) = \frac{q_{max}}{2} [1 + \sum_{r=1}^{\infty} \frac{8}{(r\pi)^2} \sin (rw_1t - \frac{\pi}{2})] \tag{4}$$

r = odd positive integers

Cosine – phase angle series representation:

$$q(t) = \frac{q_{max}}{2} [1 + \sum_{r=1}^{\infty} \frac{8}{(r\pi)^2} \cos (rw_1t - \pi)] \tag{5}$$

r = odd positive integers

Term	
1	$\frac{8}{(\pi)^2} = 0.8106$
3	$\frac{8}{(3\pi)^2} = 0.09066$
5	$\frac{8}{(5\pi)^2} = 0.0324$
7	$\frac{8}{(7\pi)^2} = 0.01654$

rw_1t (deg)	$\cos w_1t$	$\frac{8}{(\pi)^2} \cos w_1t$	$\frac{8}{(3\pi)^2} \cos 3w_1t$	$\frac{8}{(5\pi)^2} \cos 5w_1t$
0	1.000	0.8106	0.09066	0.0324
15	0.959	0.7774	0.0869	0.0311
30	0.866	0.7020	0.0785	0.0281
45	0.707	0.5731	0.0641	0.0229
60	0.500	0.4053	0.0453	0.0162
75	0.259	0.2099	0.0235	0.0084
90	0.000	0.000	0.000	0.000

Data Summary 12-3. Fourier series equations for isosceles triangle wave.

12.32. Data Summary 12-2 tabulates Fourier series data on the rectangular pulse wave, which is like the square wave except that the high and low levels are not evenly balanced over the spread of the first-order period of the wave.

FOURIER SERIES REPRESENTATION OF TRIANGULAR AND SAW TOOTH WAVES

12.33. Data Summary 12-3 gives the data for the Fourier series representation of the isosceles triangle wave. Figure 12-6 illustrates the combination of the first three components of the isosceles triangle wave.

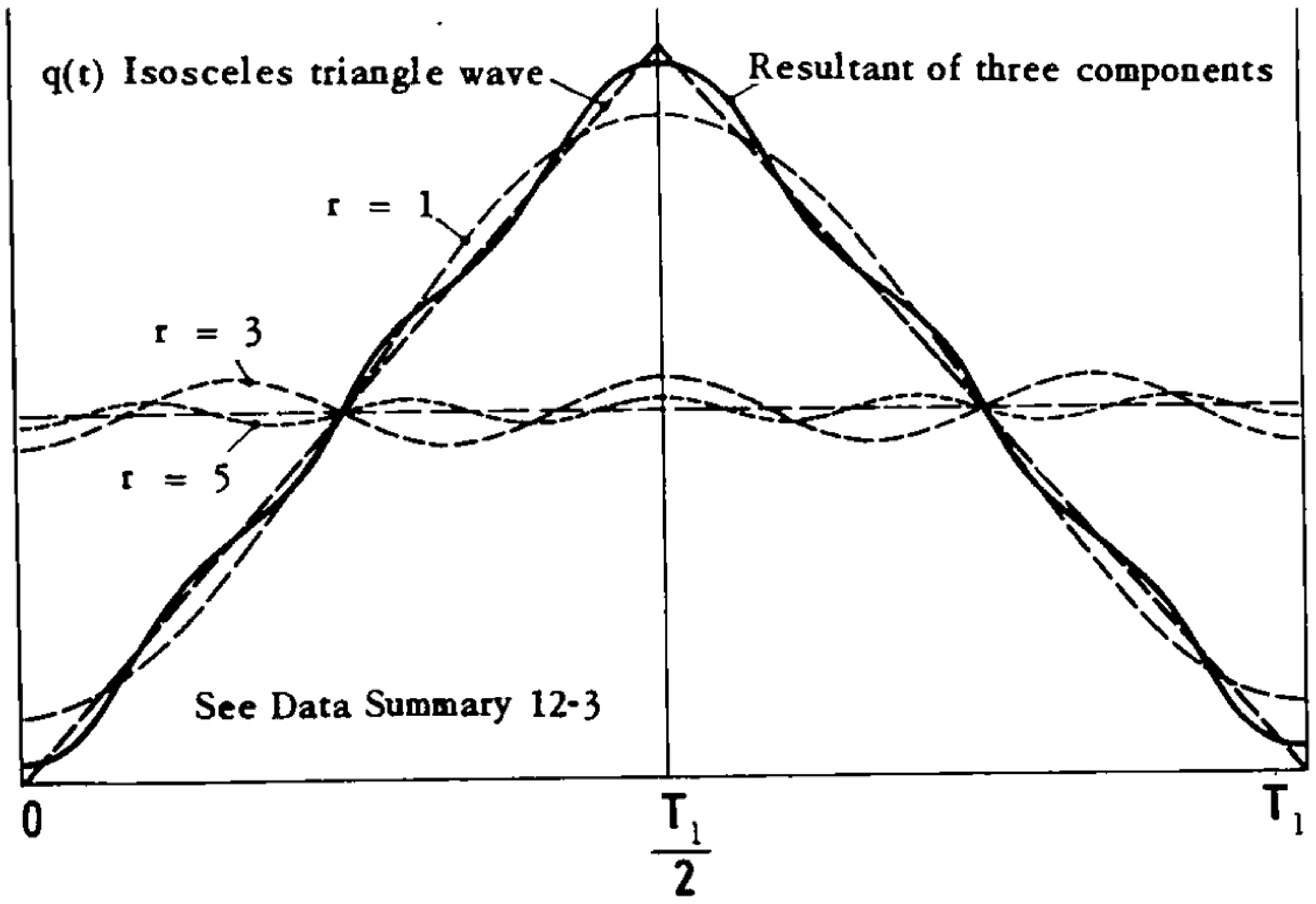


Fig. 12-6. Isosceles triangle wave approximation by first three Fourier series terms.

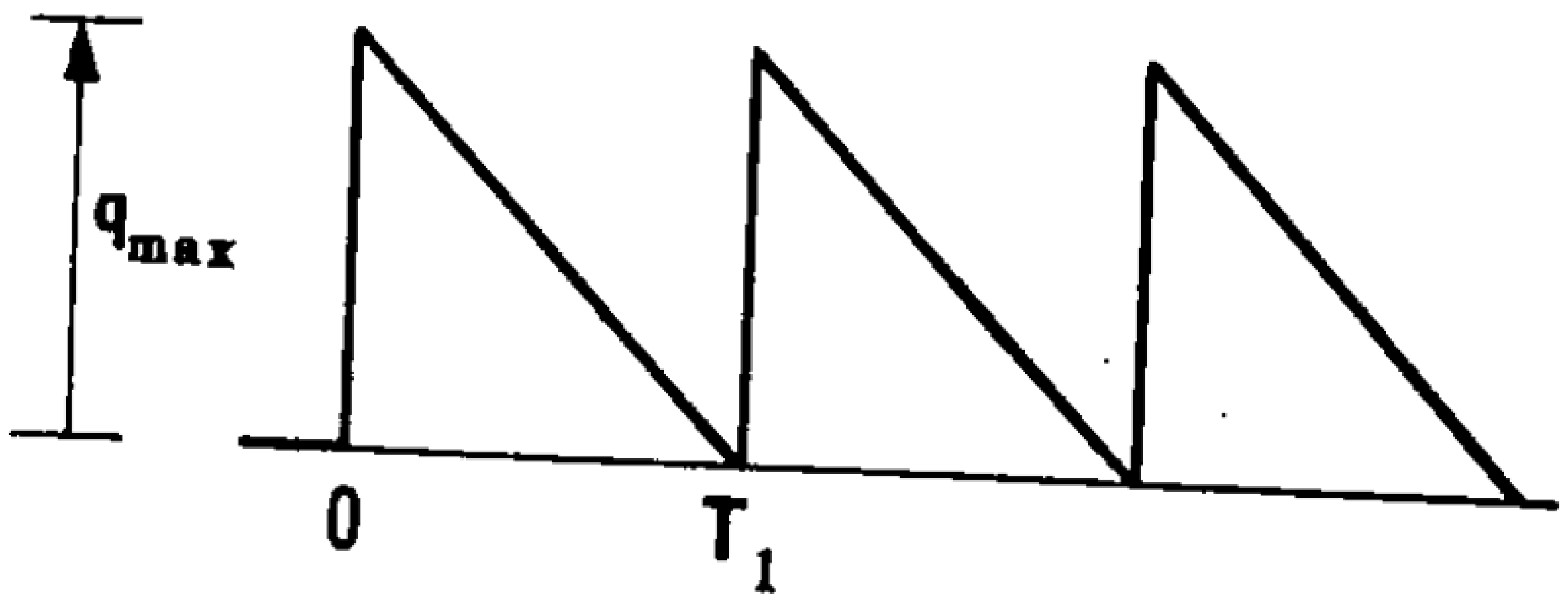
12.34. Data Summary 12-4 and Fig. 12-7 tabulate the Fourier series information for representing the saw tooth wave and show the approximation given by three sinusoidal components to the wave itself.

FOURIER REPRESENTATION OF ENGINE CYLINDER PRESSURE RECORD

12.35. The dashed-line curve of Fig. 12-8 is the experimental record of pressure inside the cylinder of an internal combustion engine. The light dashed lines represent Fourier series components found by the use of a Henrici-Coradi(26) analyzer. The resultant formed by the first six Fourier series components is shown as a heavy, full line in the figure. It is apparent that the approximation represented by even this reasonably large number of components contains relatively great deviations from the actual pressure curve.

FOURIER SERIES AMPLITUDE DISTRIBUTION PLOTS

12.36. For many purposes, it is useful to describe the Fourier series representation of any given function in terms of amplitudes for the exponential or



Definition:

$$q(t) = \frac{q_{max}}{2} \qquad t = 0, T_1$$

$$q(t) = q_{max}\left[1 - \frac{t}{T_1}\right] \qquad 0 < t < T_1 \qquad (1)$$

Exponential series representation:

$$q(t) = \frac{q_{max}}{2} \sum_{r=-\infty}^{\infty} \frac{1}{r\pi} e^{j(rw_1 t - \frac{\pi}{2})} \qquad (2)$$

Sine – cosine series representation:

$$q(t) = \frac{q_{max}}{2}\left[1 + \sum_{r=1}^{\infty} \frac{2}{r\pi} \sin rw_1 t\right] \qquad (3)$$

Sine – phase angle series representation:

$$q(t) = \frac{q_{max}}{2}\left[1 + \sum_{r=1}^{\infty} \frac{2}{r\pi} \sin rw_1 t\right] \qquad (4)$$

Cosine – phase angle series representation:

$$q(t) = \frac{q_{max}}{2}\left[1 + \sum_{r=1}^{\infty} \frac{2}{r\pi} \cos\left(rw_1 t - \frac{\pi}{2}\right)\right] \qquad (5)$$

Term		
	1	$\frac{2}{\pi} = 0.6366$
	2	$\frac{1}{\pi} = 0.3183$
	3	$\frac{2}{3\pi} = 0.2122$

rw_1t (deg)	$\sin w_1t$	$\frac{2}{\pi} \sin w_1t$	$\frac{1}{\pi} \sin 2w_1t$	$\frac{2}{3\pi} \sin 3w_1t$
0	0.000	0.0000	0.0000	0.0000
15	0.259	0.1649	0.0824	0.0550
30	0.500	0.3183	0.1592	0.1061
45	0.707	0.4501	0.2250	0.1500
60	0.866	0.5513	0.2756	0.1838
75	0.959	0.6105	0.3052	0.2035
90	1.000	0.6366	0.3183	0.2122

Data Summary 12-4. Fourier series equations for saw tooth wave.

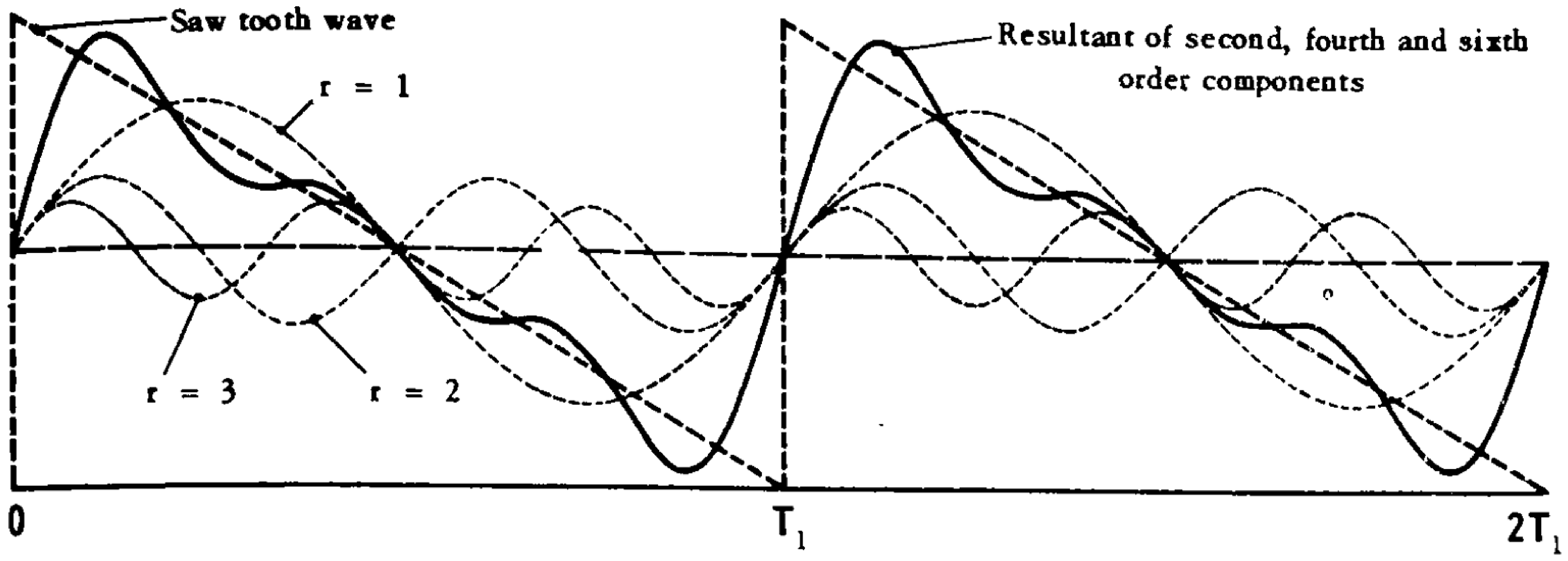
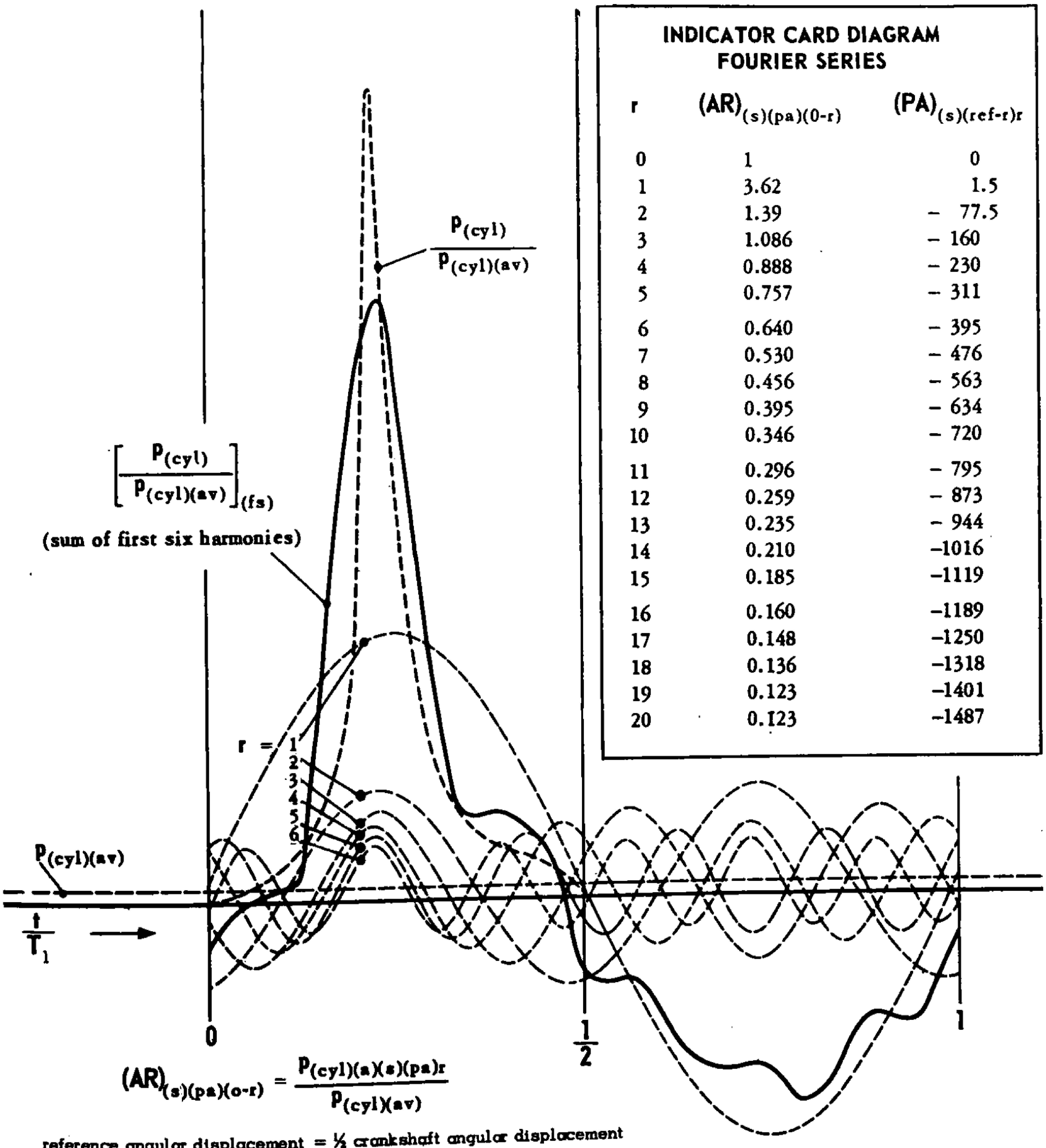


Fig. 12-7. Saw tooth wave approximation by three Fourier series terms.



INDICATOR CARD DIAGRAM FOURIER SERIES

r	$(AR)_{(s)(pa)(0\text{-}r)}$	$(PA)_{(s)(ref\text{-}r)r}$
0	1	0
1	3.62	1.5
2	1.39	− 77.5
3	1.086	− 160
4	0.888	− 230
5	0.757	− 311
6	0.640	− 395
7	0.530	− 476
8	0.456	− 563
9	0.395	− 634
10	0.346	− 720
11	0.296	− 795
12	0.259	− 873
13	0.235	− 944
14	0.210	−1016
15	0.185	−1119
16	0.160	−1189
17	0.148	−1250
18	0.136	−1318
19	0.123	−1401
20	0.123	−1487

$$(AR)_{(s)(pa)(o\text{-}r)} = \frac{P_{(cyl)(a)(s)(pa)r}}{P_{(cyl)(av)}}$$

reference angular displacement = ½ crankshaft angular displacement

Fig. 12-8. Fourier representation of engine cylinder pressure for an internal combustion engine.

sinusoidal terms. In order to simplify the representation and improve the generality of results given in plots of this type, amplitude ratios instead of amplitudes are generally used as ordinates, with component orders as abscissae. Figure 12-9 is an illustrative plot of this type for the Fourier series representation of the saw tooth wave. In this plot, the constant Fourier term is taken as the zero-order term of the series. The inherent nature of the Fourier series as a sum of discrete terms means that an amplitude ratio plot is necessarily made up of "spikes" occurring at the numbers that specify orders for the series terms. A diagram of this kind is frequently called a frequency spectrum because of its similarity to curves showing the intensities associated with the frequencies occurring in the spectrum of a light beam. Plots with discontinuous points like

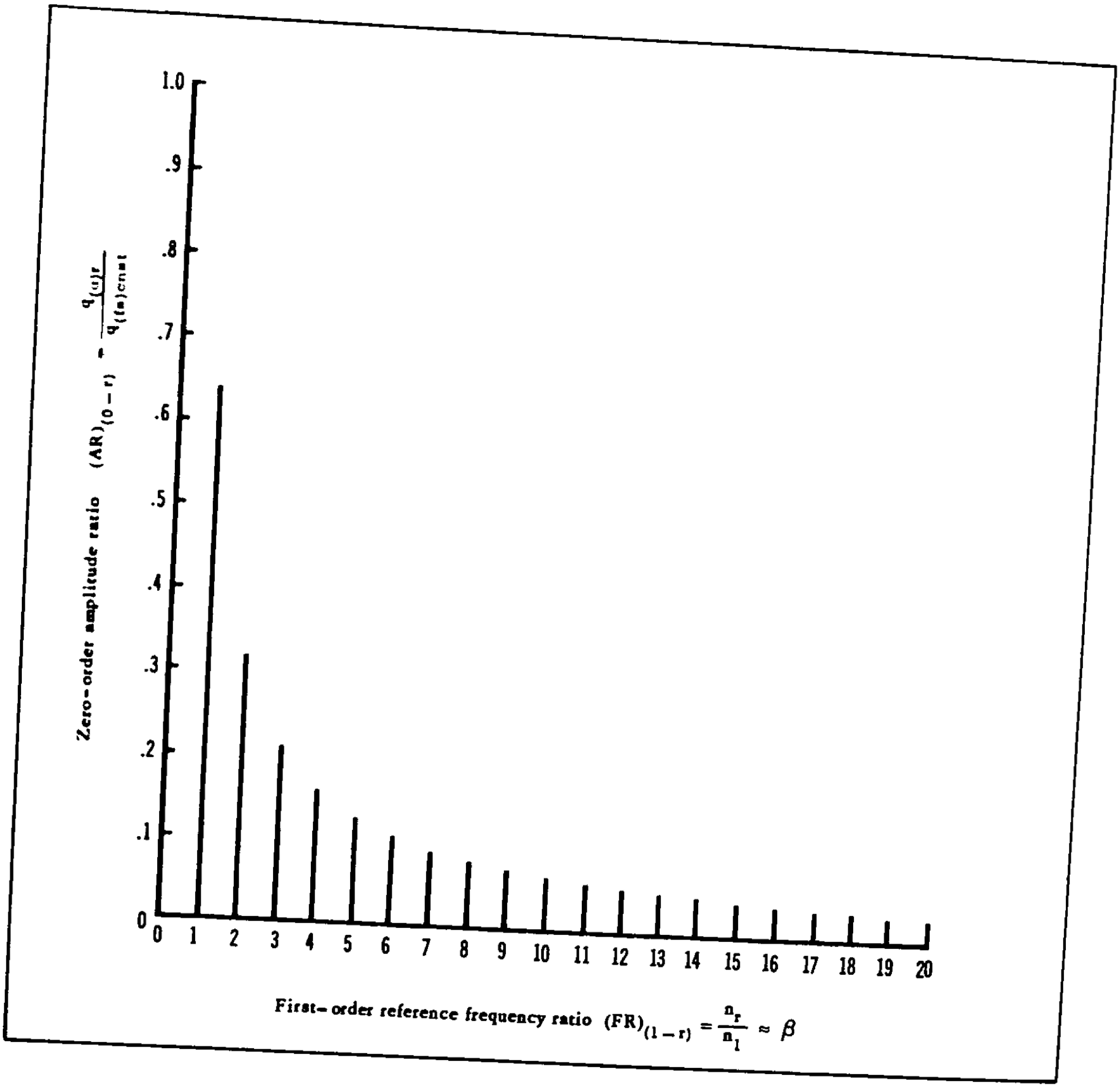
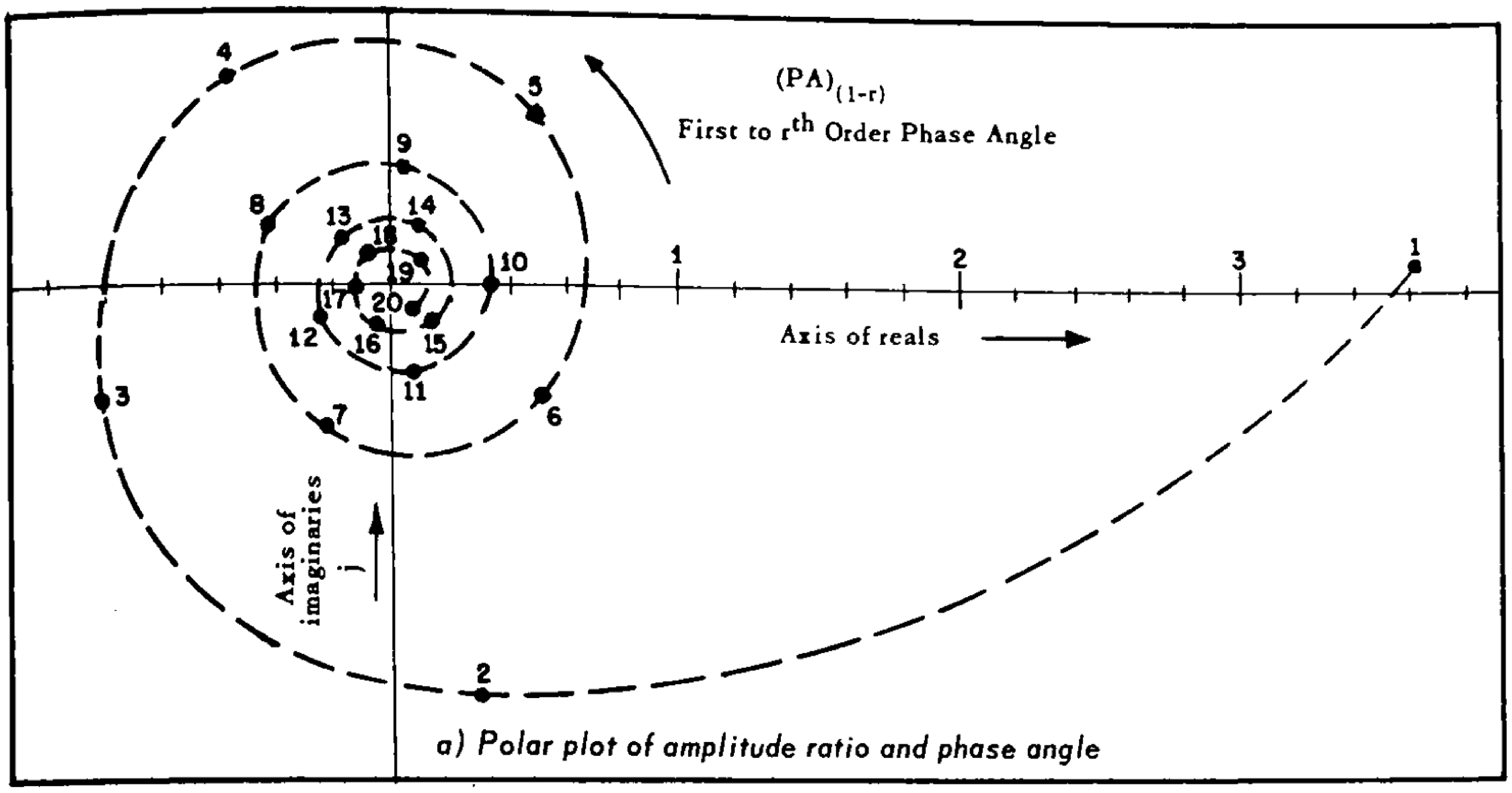
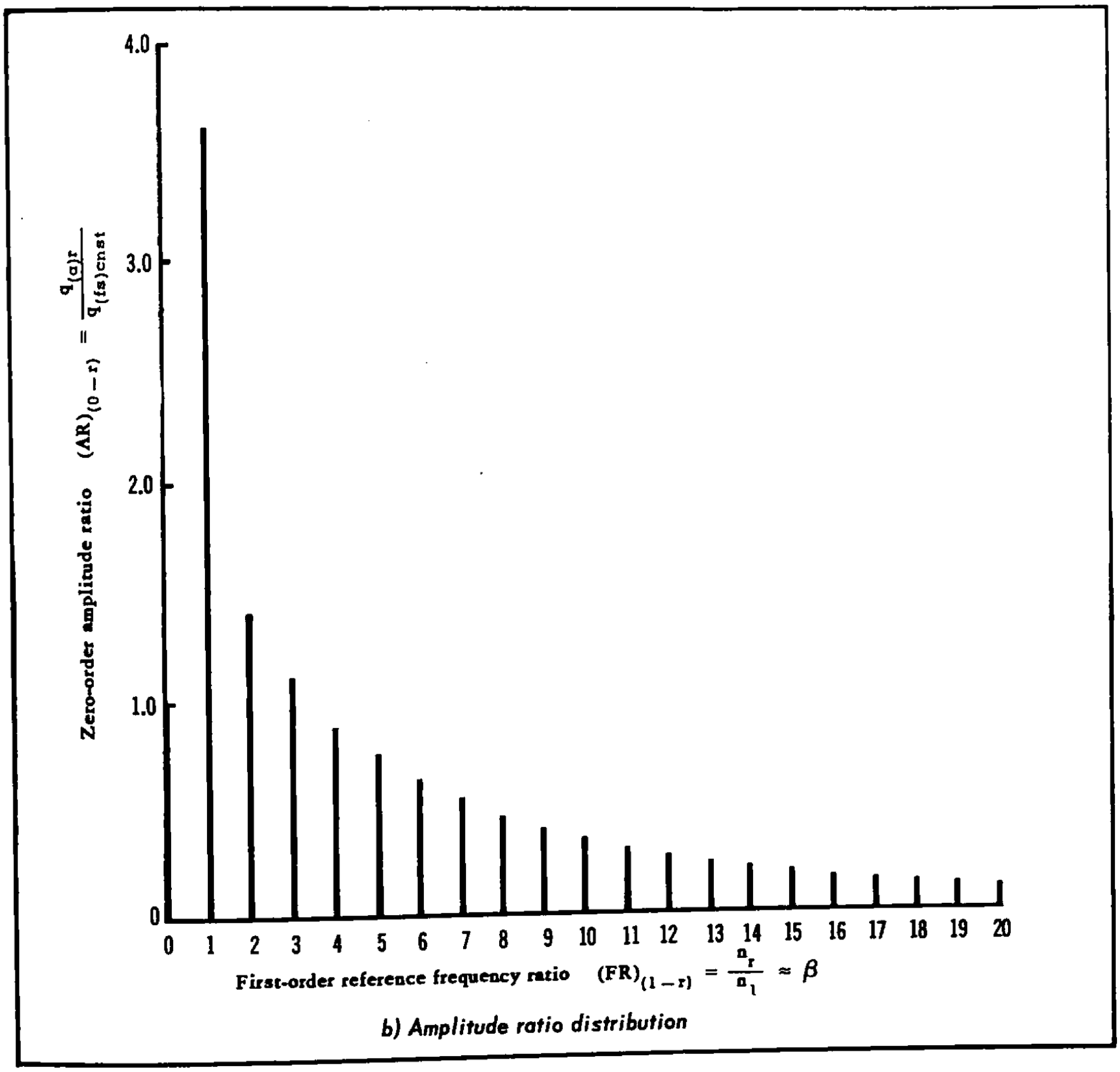


Fig. 12-9. Amplitude ratio distribution for Fourier series representation of the saw tooth wave.



a) Polar plot of amplitude ratio and phase angle



b) Amplitude ratio distribution

Fig. 12-10. Fourier series representation of internal combustion engine indicator card.

those of Fig. 12-9 correspond to a situation in which the light intensity* is concentrated in discrete frequencies to give a line spectrum. When a continuous distribution of frequencies is present, the line spectrum becomes a band spectrum for which the intensity distribution might properly be represented by a continuous curve plotted as a function of frequency rather than the component order of Fourier series terms.

12.37. Figure 12-10 is an amplitude ratio distribution curve for the Fourier series representation of the indicator card shown in Fig. 12-8. It is interesting to note in this plot that the zero-order amplitude ratio which corresponds to the average pressure level is much less than the first-order amplitude. Beyond the first-order term, the amplitude ratios decrease with a fairly smooth exponential-like trend, approaching zero asymptotically. Plots like that of Fig. 12-10 do not completely specify a Fourier series because they do not provide phase angle information. Both amplitude and phase angle information are given for the indicator card example by the polar plot of Fig. 12-10, in which the length of the vector from the origin to each point represents the amplitude ratio, and the angle of the vector from the positive direction of the axis of reals represents the phase angle for the Fourier component identified by the point.

FREQUENCY DOMAIN AND TIME DOMAIN REPRESENTATIONS FOR VARIABLES

12.38. The discussion of the preceding paragraphs shows that the essential information for describing the behavior of a variable, as a function of time, may be contained in information on amplitude ratio and phase angle as functions of the order of the Fourier series components. These plots must be supplemented by data on the magnitude of the zero-order term and the first-order period. With this information available, the time function may then be completely described by any one of the four conventional Fourier series forms of Definition Summary 12-4. In general, amplitude ratio and phase angle distribution curves with the necessary associated information specify a function in the frequency domain. The frequency domain data determine a Fourier series form that represents the given function as a function of time, that is, in the time domain. The methods that are available for carrying out transformations between the frequency domain and the time domain are among the most powerful tools available to the instrument engineer.

CONTINUOUS FREQUENCY DESCRIPTIONS OF QUANTITY VARIATIONS

12.39. The Fourier series is particularly adapted to describing functions that are periodic and may be reasonably well represented by the combination of a few sinusoidal components. However, situations often occur in nature that are difficult or impossible to represent satisfactorily by means of Fourier series terms. In particular, the components present in natural variations are very often not present as integer multiples of a first-order frequency, but include all the frequencies of a continuous band. When a continuous frequency band is present in the variation of a quantity with time, amplitude ratio and phase angle will in general both vary continuously as a function of frequency. These variations may be represented on the complex plane by plots like that illustrated in section a) of Definition Summary 12-6. The length of the vector drawn to a typical point of the plot represents the amplitude corresponding to the frequency marked on the curve for the given point. The angle between the positive real axis and the vector is the phase angle for the given frequency.

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* As the term is used here, light intensity is a measure of the rate of energy flow, i.e., the power associated with the light beam.

12.40. For comparison purposes, the set of discrete points shown on the complex plane of the diagram in section b) of Definition Summary 12-6 gives the amplitude and phase angle for the terms of an assumed Fourier series. The frequency for each point is given as the product of the order corresponding to each point and the angular frequency of the first-order term. The points may also be identified by frequency ratios defined as the ratio between the frequency for a given point and the first-order frequency. In the illustrative example, the frequency ratio (symbol either FR or the Greek letter beta (β)) for any point is simply the order of the term corresponding to the point.

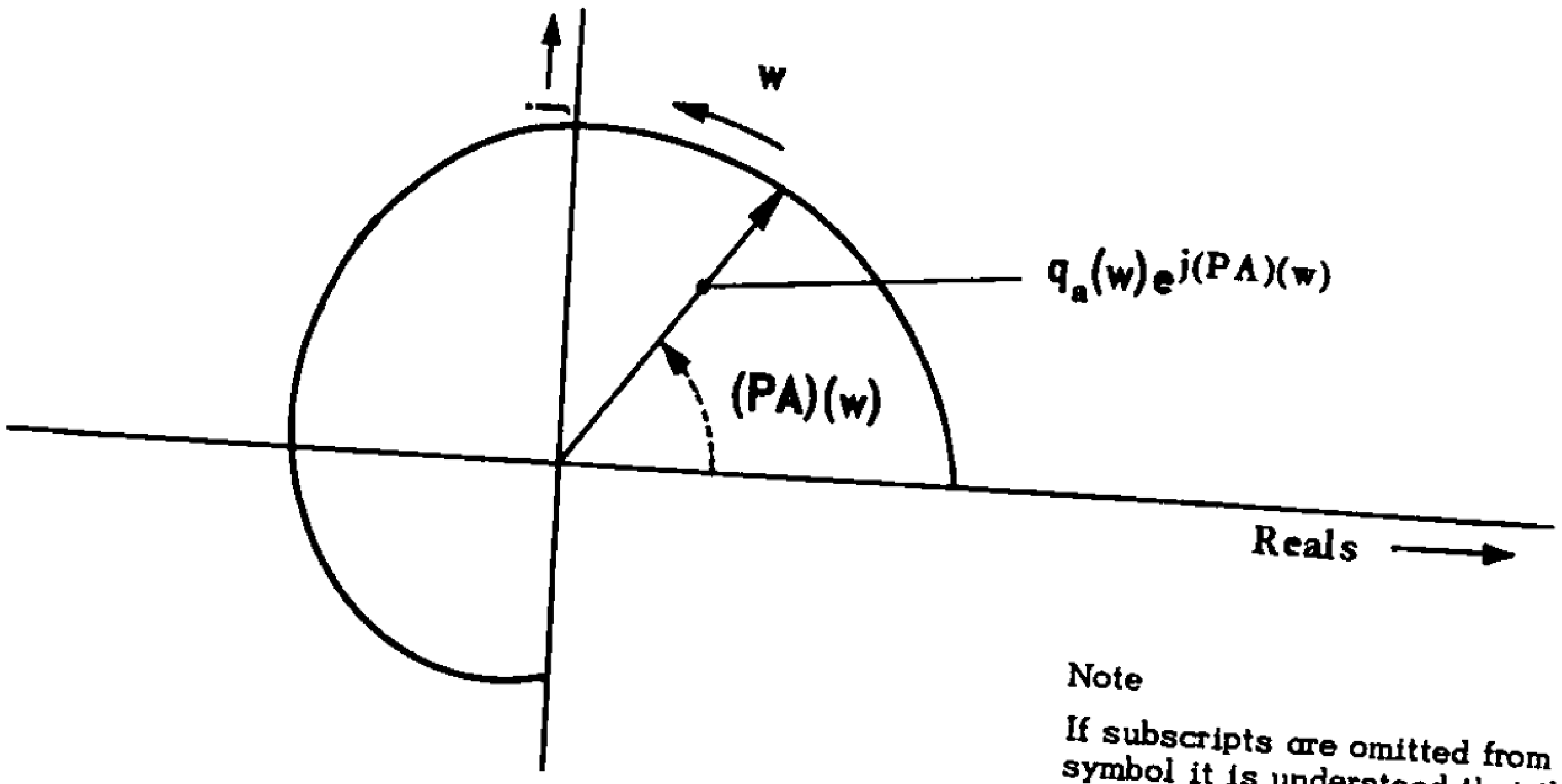
NONDIMENSIONAL REPRESENTATION OF CONTINUOUS FREQUENCY DISTRIBUTIONS

12.41. The diagram in section c) of Definition Summary 12-6 shows an illustrative plot for a continuous frequency distribution with the frequency variable expressed as the frequency ratio and the length of the vector representing an amplitude shown in terms of the amplitude ratio. For plots of this type, the reference frequency for forming the frequency ratio and the reference amplitude for forming the amplitude ratio are both arbitrary. In practice, values are chosen for these references that give numbers which vary within convenient ranges for the situations they are chosen to describe.

FOURIER INTEGRAL AND FOURIER TRANSFORMATION

12.42. Fourier series representations of variations in the time domain do not directly allow for the representation of continuous frequency bands. However, it is possible by analogy with the Fourier series form to set up an integral representation for quantities that require continuous frequency distributions to represent their variations with time. This representation is the Fourier integral. Derivations of the Fourier integral form with all its possibilities and limitations are given in a number of mathematical treatises.(8),(18),(33),(35) These derivations generally stress the niceties of mathematical reasoning, but do not concisely present the information adapted for engineering use of the Fourier integral. Derivation Summary 12-1 outlines a development of the Fourier integral in non-dimensional terms that depends upon plausibility and the equivalence of the final equations with accepted Fourier transformations as proof that the results are valid within the known limitations of these forms.

Certain functions of time may be expressed in terms of an imaginary exponential form with amplitude and phase angle varying as continuous functions of the exponential frequency. A typical plot giving amplitude and phase angle for a function of this kind is illustrated below

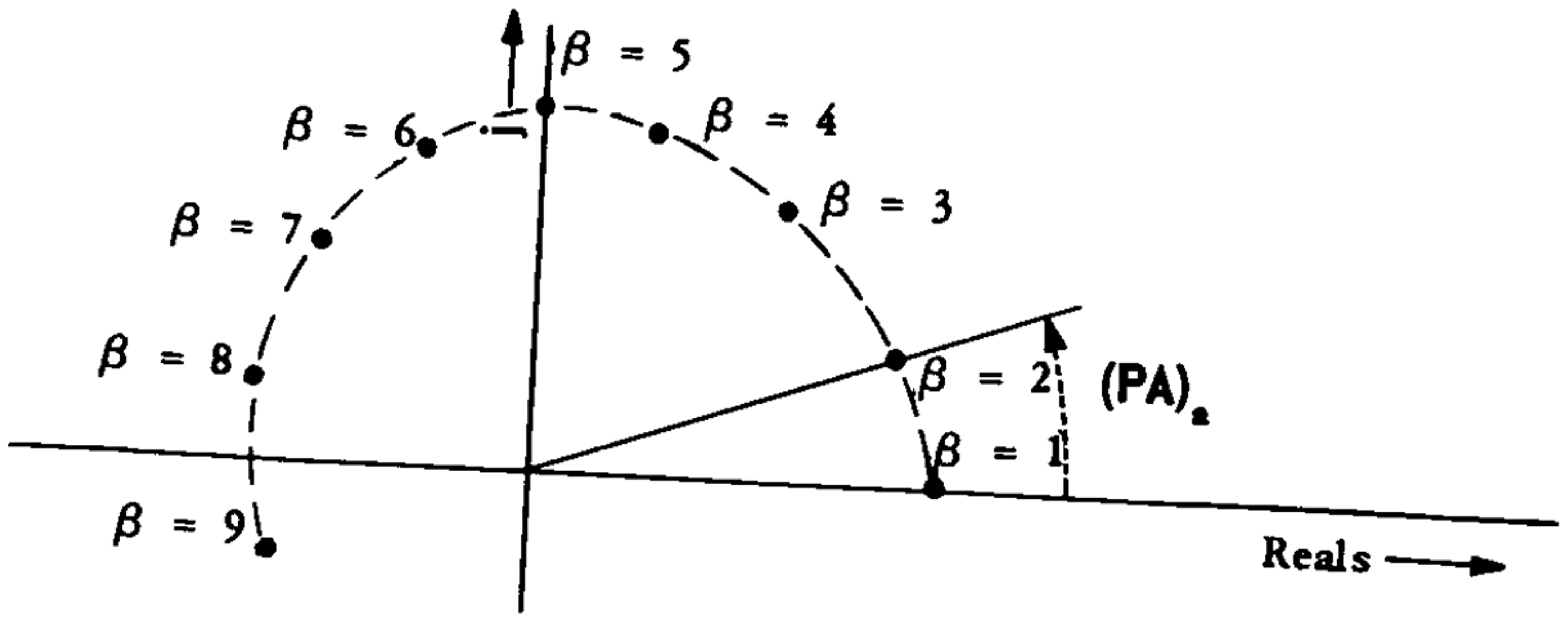


Note

If subscripts are omitted from the phase angle symbol it is understood that the phase angle is measured from the positive real axis direction.

a) Illustrative complex plane plot for continuous variation of amplitude and phase angle of an imaginary exponential representing a function of time

For comparison purposes, a plot of phase angle and amplitude for the Fourier series would be a set of unconnected dots, i.e.,



By definition

$$(FR)_{(1-r)} = \frac{w_r}{w_1} = \frac{n_r}{n_1} \approx \beta_{(1-r)}$$

= First order – r^{th} order frequency ratio

In the present illustration no possibility of misunderstanding exists, so that $\beta_{(1-r)} \rightarrow \beta$ = frequency ratio

b) Illustrative discontinuous amplitude and phase angle plot for Fourier series

Definition Summary 12-6. Continuous amplitude ratio and phase angle plots. (Page 1 of 2)

If the amplitudes of Definition Summary 12-6a are divided by a reference amplitude $q_{(a)ref}$, and the angular frequency w is divided by a reference angular frequency w_{ref}, the amplitude as a function of frequency is

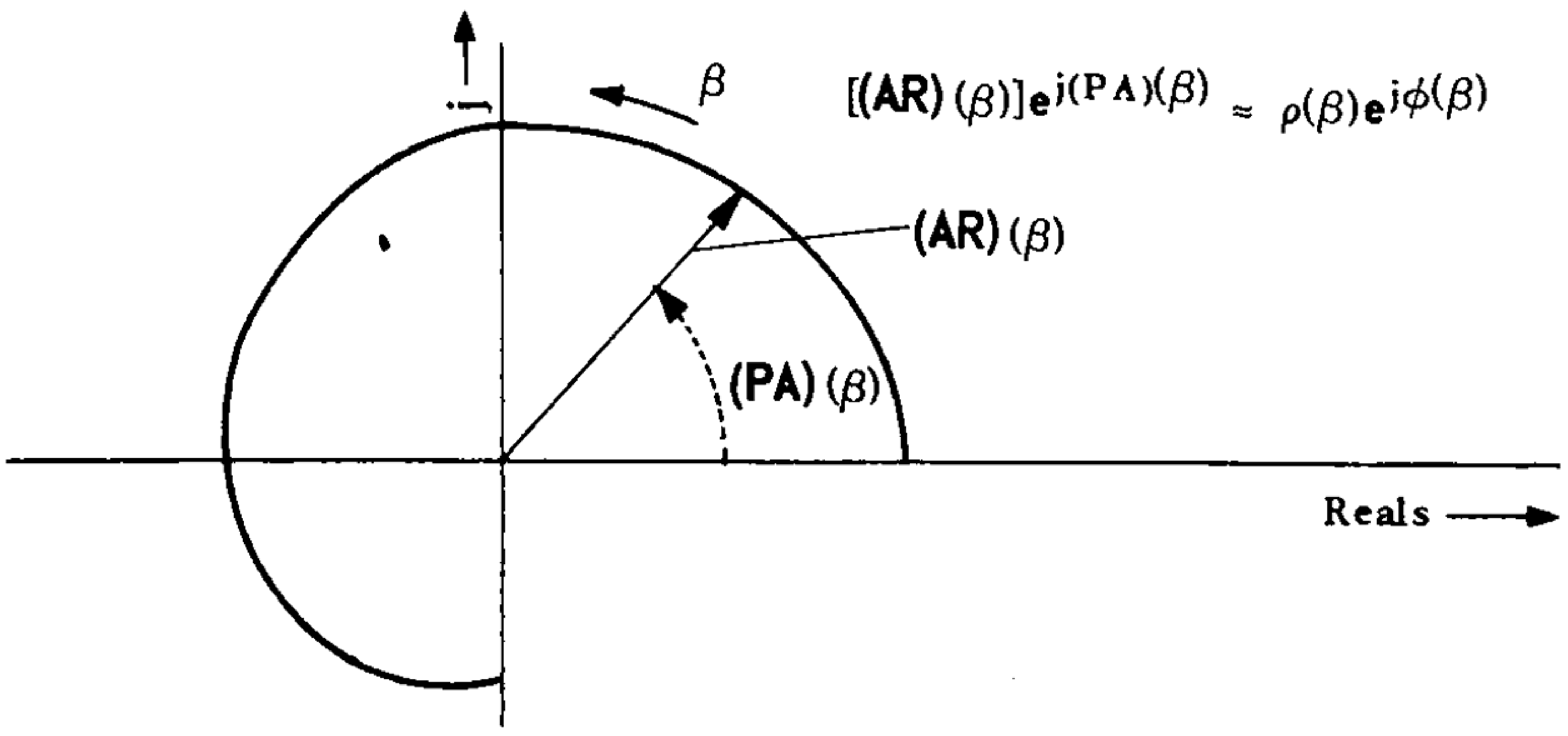
$$(AR)(w) = \rho = \frac{q_a(w)}{q_{(a)ref}} = \text{amplitude ratio} \tag{1}$$

and

$$(FR) = \beta = \frac{w}{w_{ref}} = \frac{n}{n_{ref}} = \frac{T_{ref}}{T} = \text{frequency ratio} \tag{2}$$

For these definitions the reference amplitude $q_{(a)ref}$ and the reference angular frequency may be arbitrarily chosen to have any convenient values.

In terms of these nondimensional variables, the amplitude – phase angle frequency plot of Definition Summary 12-6a becomes the amplitude ratio and phase angle – frequency ratio plot shown below.


β
$[(AR)(\beta)]e^{j(PA)(\beta)} \approx \rho(\beta)e^{j\phi(\beta)}$
(AR)(β)
(PA)(β)
Reals →

c) Illustrative continuous amplitude ratio and phase angle – frequency ratio plot

Definition Summary 12-6. Continuous amplitude ratio and phase angle plots. (Page 2 of 2)

12.43. If a function q of the time – reference period ratio, $t/T_{(ref)}$, is to depend upon a continuous band of the frequency ratio, an infinitesimally small change in the frequency ratio must produce a small change in the function $q(t/T_{(ref)})$. Let the small change in the frequency ratio, $d\beta$, occur at a given value of the frequency ratio, β, and a given value of the time – reference period ratio $t/T_{(ref)}$. The function relating the change in $q(t/T_{(ref)})$ to the small change in β is given in Eq. (1) of Derivation Summary 12-1. It consists of the product of a coefficient which is a function of β and an imaginary exponential whose exponent is the frequency ratio multiplied by $2\pi(t/T_{(ref)})$. The coefficient is a complex number expressed as a magnitude, which is an amplitude multiplied by an imaginary exponential whose exponent is a phase angle. The amplitude factor in this equation may be replaced by the amplitude ratio considered as a function of β if both sides of the equation are divided by $q_{(a)(ref)}$, an arbitrary reference amplitude.

12.44. Integration of Eq. (1) of Derivation Summary 12-1 over all values of frequency ratio from minus infinity to plus infinity gives the expression of Eq. (3)

When the amplitude and phase angle are continuous functions of frequency ratio, and q is a function of the time-reference period ratio, $\frac{t}{T_{ref}}$, the change in $q\left(\frac{t}{T_{ref}}\right)$, corresponding to an infinitesimal increment in the frequency ratio $d(\beta)$ is

$$d\left[q\left(\frac{t}{T_{ref}}\right)\right] = \{q_{(a)}(\beta)e^{j(PA)(\beta)}\}e^{j\beta 2\pi\left(\frac{t}{T_{ref}}\right)}d(\beta) \tag{1}$$

where

T_{ref} = Reference period (arbitrarily chosen)

$\beta = \frac{w}{w_{ref}}$ = Frequency ratio

w_{ref} = Reference angular frequency $= \frac{2\pi}{T_{ref}}$

This expression may be nondimensionalized by multiplying and dividing the right-hand side of this equation by $q_{(a)ref}$, i.e.,

$$\frac{d\left[q\left(\frac{t}{T_{ref}}\right)\right]}{q_{(a)ref}} = \{(AR)(\beta)e^{j(PA)(\beta)}\}e^{j\beta 2\pi\left(\frac{t}{T_{ref}}\right)}d(\beta) \tag{2}$$

$(AR)(\beta)$ = Amplitude ratio as a function of frequency ratio

Integration over the entire frequency ratio range from $-\infty$ to $+\infty$ gives

$$\frac{q\left(\frac{t}{T_{ref}}\right)}{q_{(a)ref}} = \int_{-\infty}^{+\infty}\{(AR)(\beta)e^{j(PA)(\beta)}\}e^{j\beta 2\pi\left(\frac{t}{T_{ref}}\right)}d(\beta) \tag{3}$$

This equation corresponds to the exponential form of the Fourier series with the summation of discrete frequency terms replaced by the integral based on continuously varying frequency.

The expressions required to find the amplitude ratio – phase angle exponential product factor of Eq. (3) correspond to the integrals for the amplitude ratio – phase angle exponential products of the Fourier series exponential forms (refer to Eqs. (1), (2), and (5) of Definition Summary 12-4). The derivation of these expressions follows:

By the theorem of the mean (17)

$$d\left[q\left(\frac{t}{T_{ref}}\right)\right] = q\left(\frac{t}{T_{ref}}\right)d\left(\frac{t}{T_{ref}}\right) \tag{4}$$

and

$$d[(AR)(\beta)e^{j(PA)(\beta)}] = (AR)(\beta)e^{j(PA)(\beta)}d(\beta) \tag{5}$$

This means that Eq. (1) may be written in the form:

$$\frac{q\left(\frac{t}{T_{ref}}\right)}{q_{(a)ref}}d\left(\frac{t}{T_{ref}}\right) = e^{j\beta 2\pi\left(\frac{t}{T_{ref}}\right)}d[(AR)(\beta)e^{j(PA)(\beta)}] \tag{6}$$

Derivation Summary 12-1. Derivation of Fourier integral form for representing functions in terms of continuous frequency distributions. (Page 1 of 2)

Rearranging this equation and integrating with respect to $\left(\frac{t}{T_{ref}}\right)$ over the range $-\infty$ to $+\infty$ gives

$$(AR)(\beta)e^{j(PA)(\beta)} = \int_{-\infty}^{+\infty} \frac{q\left(\frac{t}{T_{ref}}\right)}{q_{(a)ref}} e^{-j\beta 2\pi\left(\frac{t}{T_{ref}}\right)} d\left(\frac{t}{T_{ref}}\right) \tag{7}$$

It is to be noted that Eq. (7) corresponds to the direct Fourier transformation, and Eq. (3) corresponds to the inverse Fourier transformation. As given by Gardner and Barnes in Transients in Linear Systems (18)

$$[FT]f(t) = F(w) = \int_{-\infty}^{+\infty} f(t)e^{-jwt}dt \tag{8}$$

$$[FT]^{-1} F(w) = f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(w)e^{jwt}dw \tag{9}$$

where

$[FT] \rightarrow$ Direct Fourier transformation

$[FT]^{-1} \rightarrow$ Inverse Fourier transformation

Mathematical considerations necessary for rigor in the derivation of direct and inverse Fourier transformations are omitted above. The validity of results is established by the similarity in form of Eqs. (3) and (7) to the inverse and direct Fourier transformations, respectively.

Equations (3) and (7) are subject to the same restrictions that apply to the Fourier transformations. These restrictions are described in mathematical texts dealing with the Fourier series and integrals.(11) (18)

Derivation Summary 12-1. Derivation of Fourier integral form for representing functions in terms of continuous frequency distributions. (Page 2 of 2)

of the derivation summary. This expression corresponds to the exponential form of the Fourier series, in that it represents a continuous summation of terms like those of a doubly infinite number of frequencies separated by infinitesimal steps. Equation (3) of Derivation Summary 12-1 also is identical in form with the inverse Fourier transformation, which is repeated in its conventional form as Eq. (8) of Derivation Summary 12-1.

12.45. Equations (4), (5), and (6) of Derivation Summary 12-1 outline the steps required to find the integral expression for the amplitude ratio — phase angle exponential product of Eq. (3) of Derivation Summary 12-1. The integral for this product corresponds to the similar expressions for the amplitude ratio — phase angle exponential products of the exponential form of the Fourier series, and is also identical in form with the direct Fourier transformation of Eq. (7) of Definition Summary 12-1. The Fourier transformations will not be discussed further here, but will appear as useful tools in the engineering applications of Volume III.

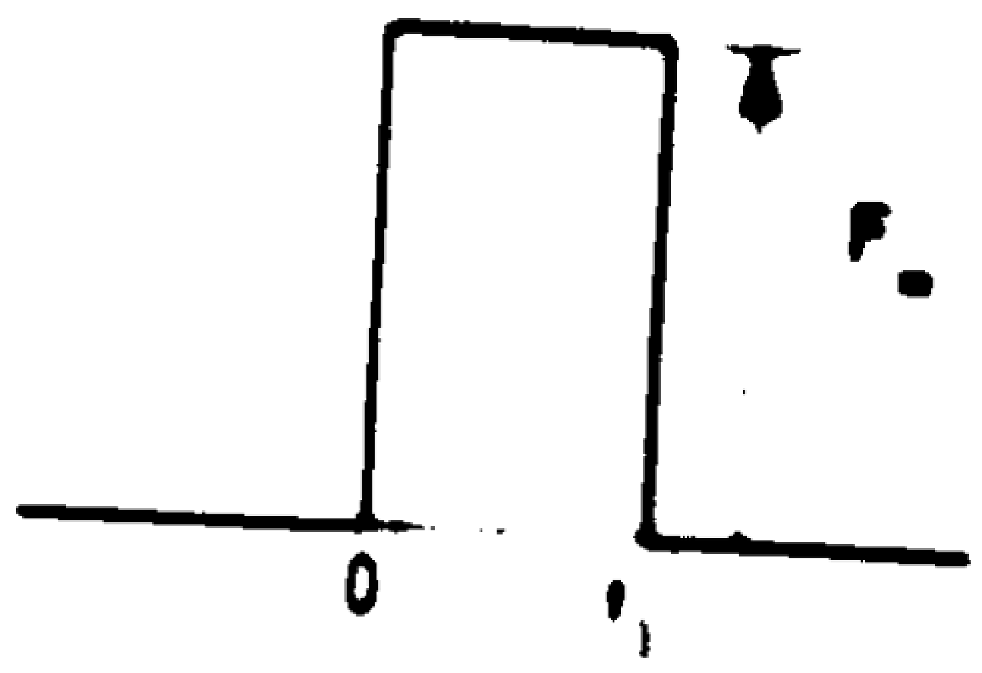
PULSE FUNCTIONS

12.46. In addition to the power series, Fourier series, and Fourier integral forms, pulse functions are generally useful for representing physical quantities. A pulse function is one which has a value different from zero for only a finite interval. Practically useful pulse functions must have finite average values. In any given situation, the shape of the pulse should be chosen to suit the variation to be described. Pulses of many types are possible. Section a) of Definition Summary 12-7 illustrates the rectangular and triangular pulses.

A pulse function, [PuF], is defined as one which has a value different from zero for only a finite interval. Examples of pulse functions are:

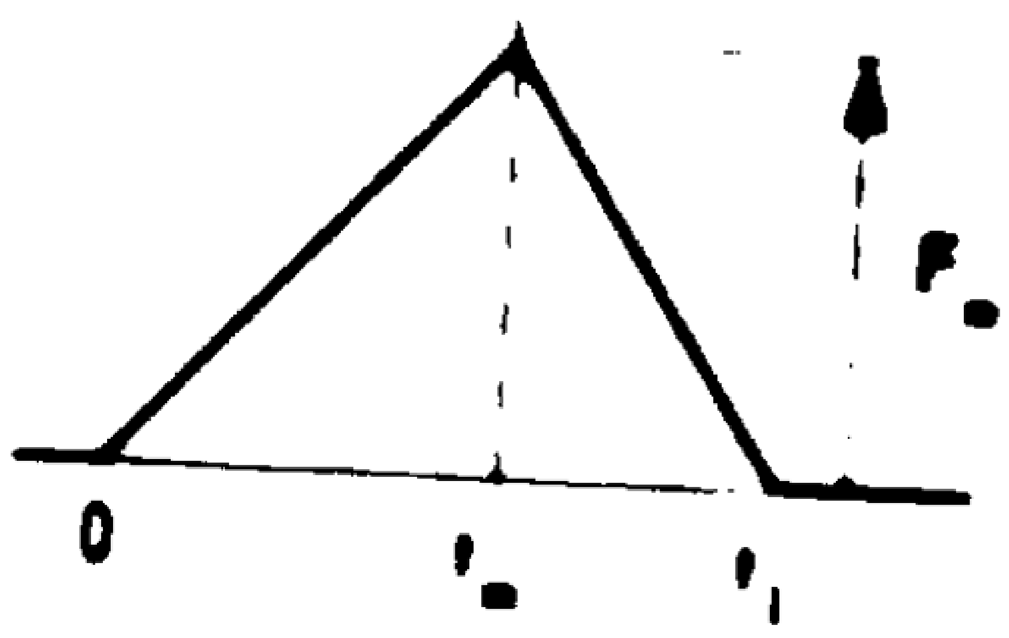
1) Rectangular pulse

$$F(t) = 0 \qquad t \le 0$$
$$F(t) = F_m = \text{constant} \qquad 0 < t < t_1 \qquad (1)$$
$$F(t) = 0 \qquad t_1 \le t$$



2) Triangular pulse

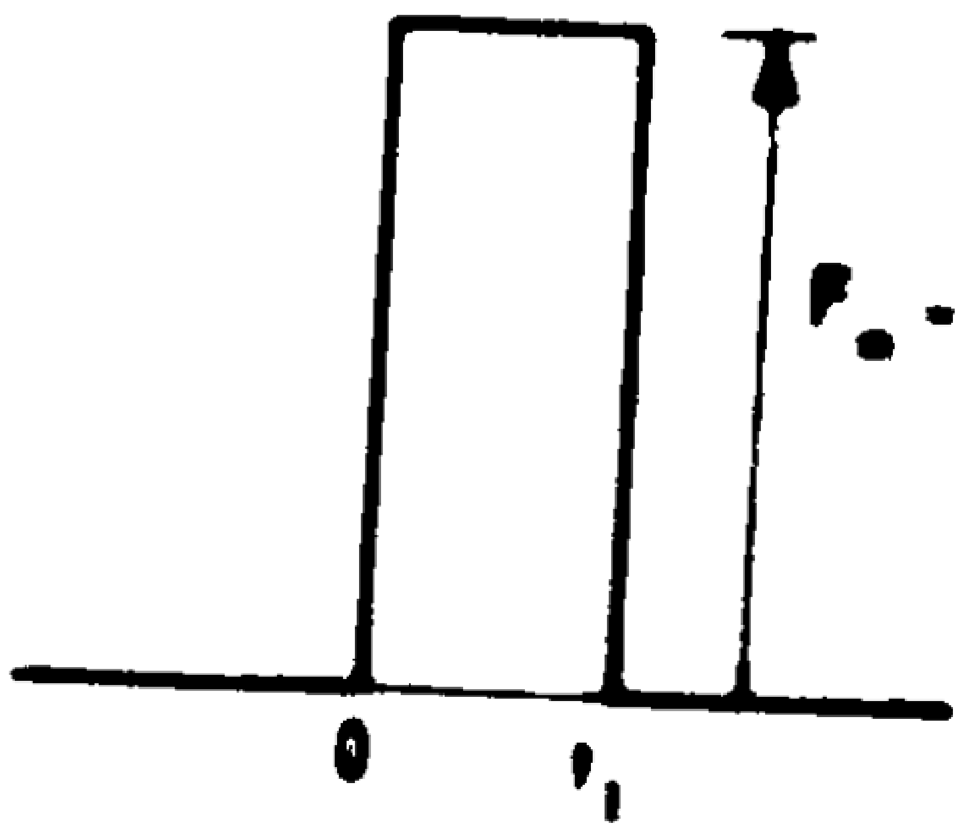
$$F(t) = 0 \qquad t \le 0$$
$$F(t) = \frac{t}{t_m} F_m \qquad 0 \le t \le t_m$$
$$F(t) = \frac{t_1 - t}{t_1 - t_m} F_m \qquad t_m \le t \le t_1 \qquad (2)$$
$$F(t) = 0 \qquad t_1 < t$$



a) Pulse functions

A time pulse function, [TPF], is one which has the dimension of inverse time. If F_m is $1/t_1$, the rectangular pulse becomes a rectangular time pulse function. A unit time pulse function, [UTPF], is one whose integral over time is equal to unity.

$$\int_{-\infty}^{\infty} [\mathrm{UTPF}]\,dt = 1 \qquad (3)$$



A rectangular pulse becomes a unit rectangle time pulse function when $F_m = 1/t_1$. The integral of a time pulse function is dimensionless.

b) Time pulse functions

Any pulse function may be represented by the product of a coefficient and a unit time pulse function.

$$[\mathrm{PuF}] = B_p[\mathrm{UTPF}], \quad B_p = \text{pulse function coefficient} \qquad (4)$$

The integral of the pulse function with respect to time is the pulse function coefficient

$$\int_{-\infty}^{\infty} [\mathrm{PuF}]\,dt = \int_{-\infty}^{\infty} B_p[\mathrm{UTPF}]\,dt = B_p \qquad (5)$$

B_p is defined as the strength of the pulse.

c) Relation between pulse and time pulse functions

Definition Summary 12-7. Pulse and impulse functions. (Page 1 of 2)

The limiting case of the unit time pulse function, when the interval tends to zero, defines the unit time impulse function:

$$[\mathrm{UTIF}](t) = \lim_{t_1 \to 0} [\mathrm{UTPF}](t) \tag{6}$$

The unit time impulse function is also known as the Dirac Function.

The integral of the unit time impulse function with respect to time is unity:

$$\int_{-\infty}^{\infty} [\mathrm{UTIF}](t)\,dt = 1 \tag{7}$$

The integral of the unit time impulse function is dimensionless.

The working variable form of the unit time impulse function is $\delta(t)$.

The function equal to a coefficient multiplied by a unit time impulse function is defined as an impulse function:

$$[\mathrm{IF}](t) = B_{(\mathrm{imp})}[\mathrm{UTIF}](t); \qquad B_{(\mathrm{imp})} = \text{impulse function coefficient} \tag{8}$$

The integral of the impulse functions with respect to time is the impulse function coefficient:

$$\int_{-\infty}^{\infty} [\mathrm{IF}](t)\,dt = \int_{-\infty}^{\infty} B_{(\mathrm{imp})}[\mathrm{UTIF}](t)\,dt = B_{(\mathrm{imp})} \tag{9}$$

$B_{(\mathrm{imp})}$ is defined as the strength of the impulse

d) Impulse functions

If $F(t)$ is a function which changes very slowly during the interval $0 < t < t_1$, and $[F(t_1)]_{(\mathrm{av})}$ is taken as the average value of the function in the interval, then

$$\int_{-\infty}^{\infty} F(t)\,[\mathrm{UTPF}]\,dt \approx [F(t_1)]_{(\mathrm{av})} \tag{10}$$

The limiting case of this integral, when the interval tends to zero, is

$$\int_{-\infty}^{\infty} F(t)\,[\mathrm{UTIF}]\,dt = F(0) \tag{11}$$

e) Integral of function – time impulse function product

Definition Summary 12-7. Pulse and impulse functions. (Page 2 of 2)

12.47. A special form of the pulse function is the time pulse function which, as defined in section b) of Definition Summary 12-7, is a pulse function which has the dimensions of inverse time. The integral of a time pulse function with respect to time has no dimensions. A unit time pulse function is one whose integral with respect to time is unity, when the limits of integration include the whole non-zero portion of the function. Any other type of pulse function can be constructed from a unit time pulse function by multiplying by an appropriate coefficient, which is defined as the strength of the pulse function in section c).

12.48. The impulse function is a mathematical abstraction derived from the pulse function concept. The limiting case of the unit time pulse function, when the duration of the non-zero interval tends to zero, defines the unit time impulse function.* The integral of the unit time impulse function with respect to time is unity, if the limits of integration include the instant of the impulse, and the integral is without dimensions. Any other impulse function may be considered as a constant factor multiplied by the unit time impulse function. In section d) of Definition Summary 12-7 this constant is defined as the strength of the impulse.

12.49. Summations of pulse functions may be used to represent continuous functions of time. The background for this representation is illustrated in Fig. 12-11. A series of rectangles of finite width will represent a continuous function of time in the sense that the step-like forms of the rectangle tops approximate the curve, and the sum of the areas of individual rectangles approaches the area under the continuous curve. The rectangle approximation becomes better as more rectangles with smaller time intervals are used in the summation. Each rectangle of this type may be represented by a rectangular pulse function whose strength is equal to the area of the rectangle. An equivalent representation is obtained by the use of an impulse function, whose strength is the area of the rectangle, applied during the interval of the rectangle. The integral of the impulse function has the value of the area of the rectangle so that the sum of the integrals of the successive impulse functions is equal to the sum of the areas of the rectangle. The limits of integration of the impulse function integrals may be decreased so long as the interval of integration includes the instant of the impulse. Interchanging the order of application of the summation and integration processes, as in Eq. (4) and Eq. (5) of Fig. 12-11, shows that the given continuous function can be represented as a sum of impulse functions. The result is given by Eq. (6). It is obvious that the area approximation becomes better as the number of impulses within a given range of integration is increased.

12.50. Continuous functions may be approximated by pulse functions with many geometrical shapes other than the rectangle. The principles involved are alike for shapes of all kinds and are applied in several examples discussed in Volume III.

REPRESENTATION OF PHYSICAL QUANTITIES DESCRIBABLE BY CONTINUOUS FREQUENCY DISTRIBUTIONS

12.51. Many physical situations cannot be satisfactorily represented by means of the discrete frequency terms of the Fourier series, but may be effectively described by the continuous frequency distributions of the Fourier integral,

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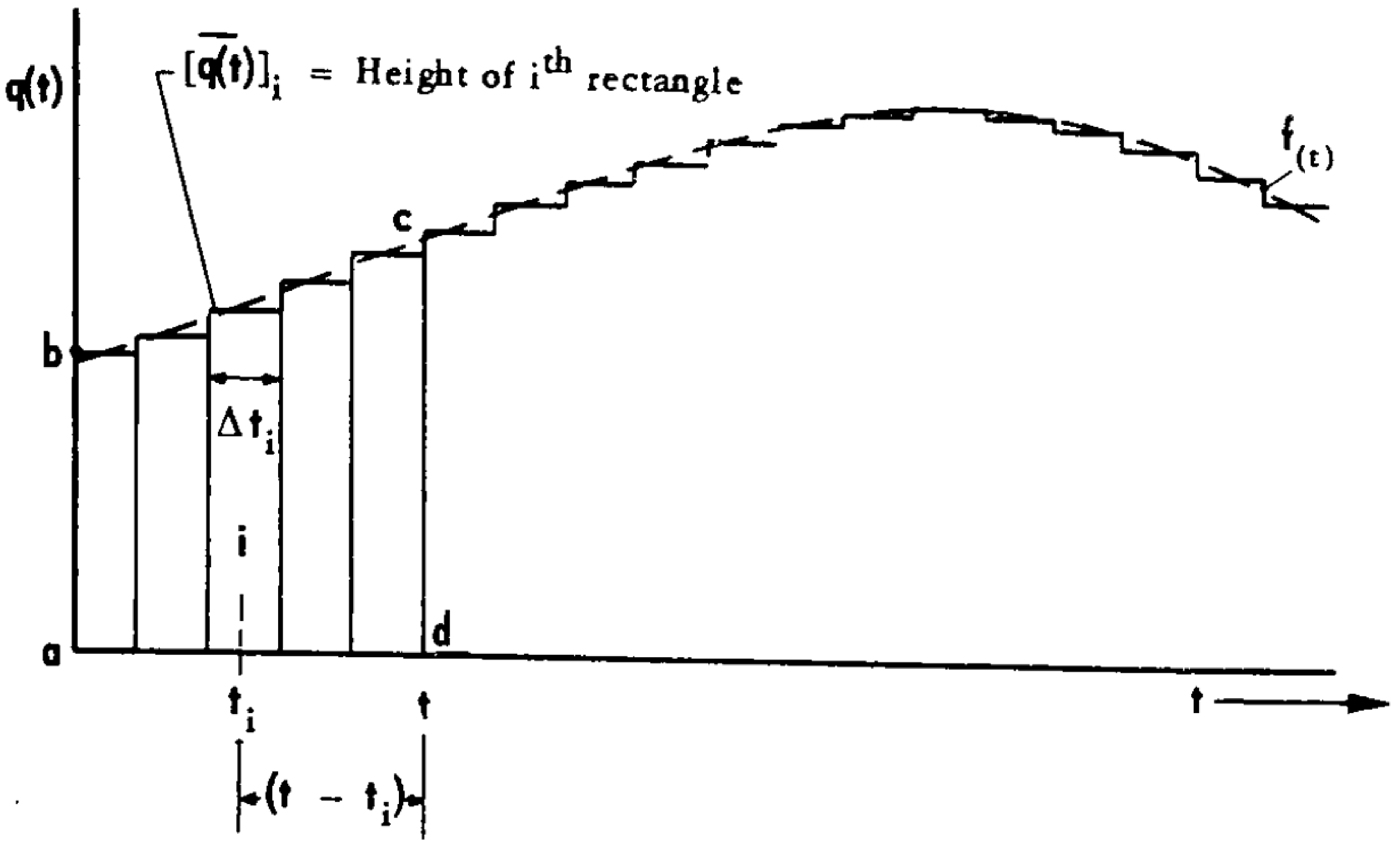
* This unit time impulse function was defined by Dirac in terms of an integral with the symbol $\delta(t)$, to describe a function which is zero if $t \neq 0$ and equal to infinity at $t = 0$, in such a way that

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

Dirac's definition of the unit time impulse function is not an ordinary function in the purely mathematical sense, but defined as the limiting case of the unit time pulse function, a relatively simple meaning can be assigned to it. See Dirac (14).



Curves representing continuous functions of time may be approximated by the tops of a series of rectangles as shown in the figure.

The area under the continuous curve between two instants of time may be approximated as a sum of areas of the rectangles, i.e.,

$$\text{area (abcd)} = \int_0^t q(t)\,dt \approx \sum_i [q(t_i)]\,\Delta t_i \tag{1}$$

The mathematical equivalent of the sum of rectangles is a sum of impulses each of which has a strength of $[q(t_i)]\,\Delta t_i$. From Definition Summary 12-7, part (d), write

$$\int_0^t q(t)\,dt \approx \sum_i \int_{-\infty}^{\infty} [q(t_i)]\,\Delta t_i\,[\mathrm{UTIF}](t - t_i)\,dt \tag{2}$$

where

$[\mathrm{UTIF}](t - t_i)$ = Unit time impulse function for the instant t_i

Note that

$$\int_{-\infty}^{\infty} [q(t_i)]\,\Delta t_i\,[\mathrm{UTIF}](t - t_i)\,dt = \int_0^t [q(t_i)]\,\Delta t_i\,[\mathrm{UTIF}](t - t_i)\,dt \tag{3}$$

provided that

$$0 < t_i < t$$

Substitute Eq. (3) in Eq. (2)

$$\int_0^t q(t)\,dt \approx \sum_i \int_0^t [q(t_i)]\,\Delta t_i\,[\mathrm{UTIF}](t - t_i)\,dt \tag{4}$$

Interchanging the summation and integration operations permits Eq. (4) to be written as

$$\int_0^t q(t)\,dt \approx \int_0^t \left\{ \sum_i [q(t_i)]\,\Delta t_i\,[\mathrm{UTIF}](t - t_i) \right\} dt \tag{5}$$

or

$$q(t) = \sum_i [q(t_i)]\,\Delta t_i\,[\mathrm{UTIF}]\,(t - t_i) \tag{6}$$

This shows that a continuous function may be represented by a sum of impulse functions.

It is useful to note that continuous functions may be represented by component forms of any convenient shape; for example, isoscles triangles may be used in the way shown above for rectangles.

Fig. 12-11. Impulse function representation for continuous functions.

discussed in paragraph 12.42. Ordinarily, problems involving continuous frequency distributions must deal with the intensity of the action described as a function of frequency rather than simply the amplitude of some quantity at each frequency. The energy, or more usually the rate of energy flow, i.e., the power associated with each frequency, is generally the variable chosen to represent intensity. In some instances the intensity measure is actually power, while in other cases the intensity measure is called power for reasons of analogy, rather than because it is physically identifiable as a rate of flow of energy. With the term given this generalized meaning, power varies as a function of frequency so that some amount of power is associated with each infinitesimally small continuous frequency band. This situation leads to the concept of power density, which is the power per unit frequency range. Because a continuous band of light waves may be spread out to give a spectrum with the distribution of intensity over frequencies as its primary characteristic, the distribution of generalized power over a frequency band is associated with a power spectral density. This quantity is rigorously defined in formal mathematical terms by Wiener(34) and Lee.(25) The object of the discussion that follows in this section is not to repeat the mathematical developments, but rather to illustrate the physical meanings of the concepts required to describe physical quantities that may be represented by continuous frequency distributions. To facilitate actual applications of continuous frequency descriptions to engineering problems, all developments are carried out in nondimensional terms with clearly defined relationships to the quantities that will usually be represented by measurements. The general method of procedure is to carry through a derivation of concepts and symbols to describe the distribution of power among the discrete frequencies of a Fourier series and then to modify these concepts so that they may be applied to situations that must be represented by continuous frequency distributions.

12.52. It is important to note that all of the performance-describing concepts associated with frequency as the independent variable may be directly applied to situations where the information is all or partly based on experimental tests. In practice, empirical data of this kind is processed by graphical methods that parallel the analytical procedures of formal mathematics. From the standpoint of engineering, results based on curves and numbers may be made to yield satisfactory results whether or not equations are used in the procedure of finding solutions.

PHYSICAL EFFECTS ASSOCIATED WITH PLANE SOUND WAVES

12.53. In order to illustrate the steps required to develop the ideas associated with power spectral density in terms of a relatively simple actual physical situation, the concepts for describing the continuous frequency distribution for a physical situation are developed by a discussion of the power flow across unit areas of a plane sound wave in a gas. The essential physical aspects of the sound problem are outlined in section a) of Derivation Summary 12-2. A detailed derivation of the wave equation for sound is omitted here because it is unnecessary for the present discussion. The basic relationship between sound wave pressure and particle motion is taken from a treatment by Fletcher.(16) Fletcher's result shows that as plane waves of sound pass through a gas, the particles of any plane oscillate about the equilibrium position the plane would have if no sound were present. At the same time, the sound is accompanied by variations of pressure from the equilibrium level that would exist in the absence of sound waves. The difference between the instantaneous pressure and the equilibrium pressure is the excess pressure (symbol p). The changes in excess pressure and the particle motions are parallel to the direction of propagation of the sound. For present purposes, the essential physical relationship is that the excess pressure is equal to the equilibrium gas density multiplied by the velocity of sound in the gas and by the particle velocity. Section b) of Derivation Summary 12-2 gives Fourier series expressions for excess pressure and particle velocity in plane sound waves of arbitrary wave shape. These two series do not include constant terms because, by definition, sound waves are only indirectly influenced by the equilibrium pressure of the undisturbed gas. The variable terms are nondimensionalized by dividing the amplitude of each term by the amplitude of the first-order term to express sinusoidal term amplitude in terms of amplitude ratios. These amplitude ratios are followed by the symbol p* to show that the amplitude ratio of the excess pressure is considered. The subscript (1-r)** for the amplitude ratio of the r^{th} term shows that the excess pressure amplitude of the first-order term is used as the reference pressure.

POWER FLOW ASSOCIATED WITH PLANE SOUND WAVES

12.54. The work done on any given plane area by the disturbances that make up the sound wave is the time integral of the volume swept out by the area, multiplied by the pressure acting to move the particles of the area. The power transferred across this area by the sound wave is the time rate of change of the work done by the sound wave. The volume swept out by motion of a given area normal to the propagation direction of the wave is equal to the area multiplied by the time integral of the particle displacements due to the wave. The rate of change of this volume is just the particle velocity produced by the wave multiplied by the area. This means that the power transferred across a given area is equal to the product of the excess pressure, the particle velocity, and the area. Section c) of

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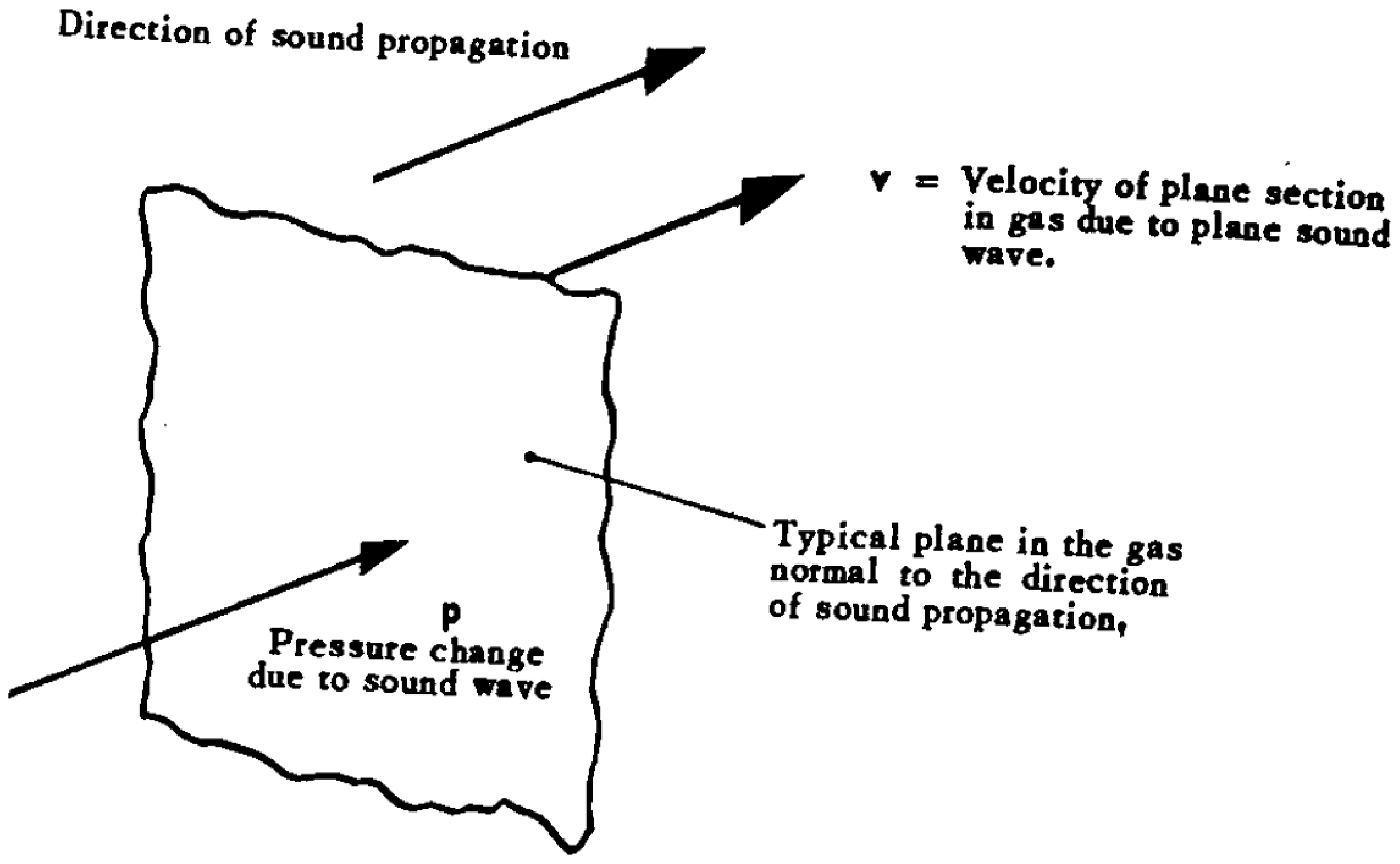
* When the amplitude ratio of some quantity other than excess pressure is considered, it is only necessary to replace p by the symbol for the new quantity.

** When a reference other than the amplitude of the first component is used, the subscript nearest the main symbol is changed to the symbol for the reference quantity.


Direction of sound propagation
v = Velocity of plane section in gas due to plane sound wave.
Typical plane in the gas normal to the direction of sound propagation,
p
Pressure change
due to sound wave

It is shown in the theory of sound (see Speech and Hearing, by Fletcher (16)) that for a plane sound wave in which the variation of pressure from its equilibrium value is small compared to the equilibrium pressure

$$p = \rho_o c v \tag{1}$$

where

ρ_o = Equilibrium density of the gas

c = Velocity of sound in the gas

v = Velocity of a given plane in the gas with respect to the no-sound-wave position of the plane

p = (Total instantaneous pressure in the plane) – (Equilibrium pressure in the gas)

= Sound wave excess pressure

a) General relationship between particle velocity and excess pressure in a plane sound wave propagated by a perfect gas

Consider a plane sound wave for which the excess pressure in a typical plane of the sound wave may be represented by a Fourier series of the form

$$p(t) = P_{(1)a} \sum_{r=1}^{r=\infty} [(AR)p]_{(1-r)} \sin \left[r 2\pi \frac{t}{T_1} + (PA)_{(1-r)r}\right] \tag{2}$$

Note that no constant term appears in this series because the equilibrium pressure is in effect the zero pressure level for the excess pressure $p(t)$.

Using the relationship of Eq. (1), the velocity of particles in a typical plane due to the action of the sound wave is

$$v(t) = \frac{p(t)}{\rho_o c} = \frac{P_{(1)a}}{\rho_o c} \sum_{r=1}^{r=\infty} [(AR)p]_{(1-r)} \sin \left[r 2\pi \frac{t}{T_1} + (PA)_{(1-r)r}\right] \tag{3}$$

b) Fourier series expressions for excess pressure and particle velocity in a plane sound wave

Derivation Summary 12-2. Power transferred by plane sound wave in terms of amplitude ratios for frequency components ; sound intensity. (Page 1 of 3)

The instantaneous power flow across a given area normal to the direction of sound propagation is given by the product of the excess pressure, the sound wave velocity of particles in the plane, and the area considered, i.e.,

$$P(t) = p(t)\,v(t)\,(\text{area}) \tag{4}$$

The power transferred by the plane wave per unit area is

$$\frac{P(t)}{(\text{area})} = p(t)\,v(t) \tag{5}$$

This means that for the wave in which the pressure and velocity variations are described by the Fourier series forms of Eqs. (2) and (3)

$$\begin{aligned}\frac{P(t)}{(\text{area})} &= \frac{p^2_{(1)a}}{\rho_o c} \sum_{r=1}^{r=\infty} [(AR)p]_{(1-r)} \sin\left[r 2\pi \frac{t}{T_1} + (PA)_{(1-r)r}\right] \\ &\quad \times \sum_{s=1}^{s=\infty} [(AR)p]_{(1-s)} \sin\left[s 2\pi \frac{t}{T_1} + (PA)_{(1-s)s}\right]\end{aligned} \tag{6}$$

c) Instantaneous power transfer across a unit area by a plane sound wave

The sound energy transferred across a unit area of the plane wave is found by integrating both sides of Eq. (6) with respect to time. The average sound power transferred during any interval of time is equal to the sound energy transferred divided by the interval. Since power is a positive scalar quantity, the total average power associated with any sound wave is the sum of the average power of each component of the sound wave. The average power of each component may be obtained by averaging the instantaneous power over the period of the first-order component.

$$\begin{aligned}\frac{P_{(av)}}{(\text{area})} &= \frac{1}{T_1}\int_0^{T_1} \frac{p(t)}{(\text{area})}\,dt = \int_0^1 \frac{p(t/T_1)}{(\text{area})}\,d\,\frac{t}{T_1} \\ &= \int_0^1 pv\,d\left(\frac{t}{T_1}\right) \\ &= \frac{p^2_{(1)a}}{\rho_o c}\int_0^1 \left\{\sum_{r=1}^{\infty} [(AR)p]_{(1-r)} \sin\left[r 2\pi \frac{t}{T_1} + (PA)_{(1-r)r}\right]\right\}^2 d\left(\frac{t}{T_1}\right) \\ &= \frac{p^2_{(1)a}}{\rho_o c}\int_0^1 \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} [(AR)p]_{(1-r)}[(AR)p]_{(1-s)} \\ &\quad \times \sin\left[r 2\pi \frac{t}{T_1} + (PA)_{(1-r)r}\right] \sin\left[s 2\pi \frac{t}{T_1} + (PA)_{(1-s)s}\right] d\left(\frac{t}{T_1}\right)\end{aligned} \tag{7}$$

Application of the relationships of Definition Summary 12-5 shows that

$$\frac{P_{(av)}}{(\text{area})} = \frac{p^2_{(1)a}}{2\rho_o c} \sum_{r=1}^{\infty} [(AR)p]^2_{(1-r)}$$

The individual r^{th} terms of the summation give the average power for the r^{th} component

$$\frac{P_{(av)r}}{(\text{area})} = \frac{p^2_{(1)a}}{2\rho_o c} [(AR)p]^2_{(1-r)} \tag{8}$$

d) Average sound power transferred across a unit area of a plane wave by the r^{th} frequency component

Derivation Summary 12-2. Power transferred by plane sound wave in terms of amplitude ratios for frequency components ; sound intensity. (Page 2 of 3)

If the coefficient of the first amplitude ratio term is taken as the reference sound power area density

$$(\mathrm{SPAD})_{ref} = \frac{p^2_{(1)a}}{2\rho_o c} \tag{9}$$

the sound power area density ratio for the r^th frequency component is

$$(\mathrm{SPADR})_{(1-r)} = \frac{P_{(av)r}}{(\text{area})(\mathrm{SPAD})_{(av)ref}} = \frac{P_{(av)r}}{(\text{area})\,\dfrac{p^2_{(1)a}}{2\rho_o c}} = [(\mathrm{AR})p]^2_{(1-r)} \tag{10}$$

It is common practice (see p. 315 of Speech and Hearing, by Fletcher (16)) to represent $\frac{P_{(av)}}{(\text{area})}$, the sound intensity (symbol (SI)), by the symbol J. Thus for the r^th sound frequency component

$$\frac{P_{(av)r}}{(\text{area})} \approx J_r = (\mathrm{SI})_r = (\mathrm{SPAD})_r$$

The sound power area density ratio becomes the r^th frequency component sound intensity ratio

$$(\mathrm{SIR})_{(1-t)} = \frac{(\mathrm{SI})_r}{(\mathrm{SI})_{ref}} = \frac{J_r}{J_{ref}} = \frac{J_r}{\dfrac{p^2_{(1)a}}{2\rho_o c}} \tag{11}$$

where

$$(\mathrm{SI})_{ref} = \frac{p^2_{(1)a}}{2\rho_o c}$$

so that

$$(\mathrm{SIR})_{(1-r)} = [(\mathrm{AR})p]^2_{(1-r)} = (\mathrm{SPADR})_{(1-r)} \tag{12}$$

e) Sound power area density ratio for r^th frequency component; sound intensity ratio for r^th frequency component

The total sound power area density ratio for any combination of Fourier series components is found by summing up the amplitude ratios for the individual frequency components, i.e.,

$$(\mathrm{SIR})_t = (\mathrm{SPADR})_t = \sum_{r=1}^{r=\infty} [(\mathrm{AR})p]^2_{(1-r)} \tag{13}$$

and

$$(\mathrm{SIR})_t = (\mathrm{SPADR})_t = \text{Total sound intensity ratio} \tag{14}$$

The total sound intensity $(\mathrm{SI})_t$, and its equivalent $(\mathrm{SPAD})_t$, the total sound power area density, are given by

$$\begin{aligned}(\mathrm{SI})_t &= (\mathrm{SPAD})_t = (\mathrm{SPADR})_t(\mathrm{SPAD})_{ref} \\ &= \frac{p^2_{(1)a}}{2\rho_o c} \sum_{r=1}^{r=\infty} [(\mathrm{AR})p]^2_{(1-r)}\end{aligned} \tag{15}$$

f) Total sound power area density; total sound intensity

Derivation Summary 12-2. Power transferred by plane sound wave in terms of amplitude ratios for frequency components; sound intensity. (Page 3 of 3)

Derivation Summary 12-2 gives this result in formal terms. The power flowing across unit area is the product of two Fourier series multiplied by a constant coefficient. This coefficient is equal to the square of the excess pressure amplitude for the first-order component* divided by the product of the equilibrium density and the velocity of sound.

FOURIER SERIES REPRESENTATION OF PLANE SOUND WAVE POWER FLOW

12.55. Section d) of Derivation Summary 12-2 shows that because sinusoidal terms have the property of orthogonality (these properties are tabulated in Definition Summary 12-5), only terms of identical order in the two Fourier series of Eq. (6) of Derivation Summary 12-2 give results that are different from zero when integration is carried out over a complete period for the first-order component. This fact makes it possible to find the average power flowing across a unit area of the plane wave due to any one frequency component. Equation (8) of Derivation Summary 12-2 shows that the power associated with the r^{th} component is equal to the square of the excess pressure for the first component, multiplied by the square of the amplitude ratio for the r^{th} term and by a constant coefficient. This coefficient is identical with that appearing in Eq. (6) of Derivation Summary 12-2.

PLANE WAVE SOUND INTENSITY

12.56. The average power flowing across a unit plane wave area due to the sound wave is the average sound power area density. This quantity is commonly called the sound intensity,** and is given the symbol J. In this discussion, the sound intensity due to the r^{th} frequency component is given the main symbol (SI) and the subscript r to show that it is associated with the r^{th} frequency component. In section e) of Derivation Summary 12-2, the sound intensity terms are nondimensionalized by a reference average sound power area density equal to the coefficient of the amplitude ratio term in Eq. (8) of Derivation Summary 12-2. This is equivalent to defining the r^{th} order sound power area density ratio in terms of the first-order sound power area density. Equation (12) of Derivation Summary 12-2 shows that the corresponding r^{th} component sound intensity ratio is equal to the squared first-to-r^{th} component excess pressure amplitude ratio.

12.57. Because the sound power area density is an always positive scalar quantity, the total sound power area density due to any sound wave is equal to the sum of the sound power area densities due to all the components. This total sound power area density, which is identical with the total sound intensity, is defined in formal terms by Eq. (15) of Derivation Summary 12-2 as the summation of all amplitude ratio terms multiplied by the coefficient of Eq. (6) of Derivation Summary 12-2. Section f) of Derivation Summary 12-2 defines the sound intensity ratio as the sum of the sound intensity ratios for the individual components. The corresponding total sound intensity is equal to the total sound power area density ratio multiplied by the reference sound power area density.

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* This pressure is not necessarily the amplitude of the first-order term; it may be the amplitude of any other term, or it may be any pressure value that is chosen as the reference pressure.

** See Appendix B of Speech and Hearing by Fletcher (16).

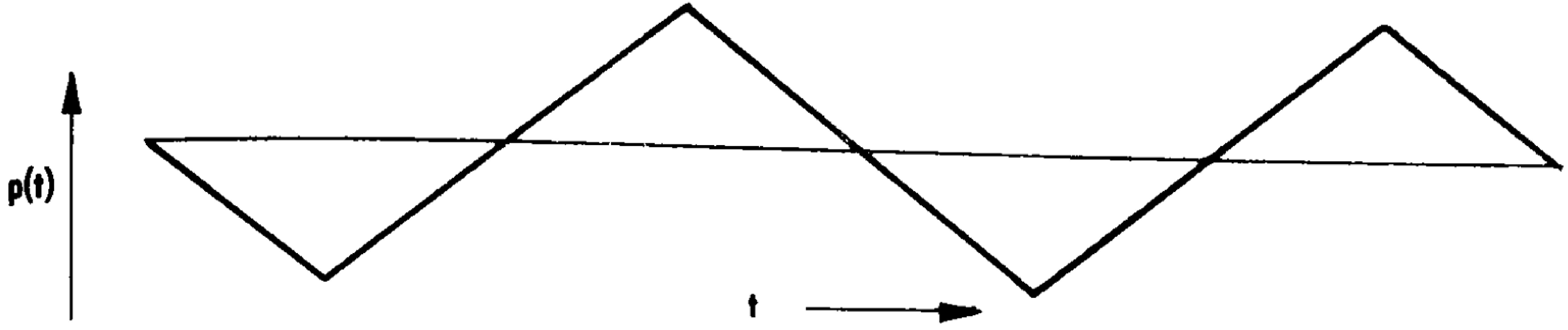
PLANE SOUND WAVE DISCONTINUOUS FREQUENCY POWER DISTRIBUTION

12.58. Derivation Summary 12-3 describes an application of the concepts developed in Derivation Summary 12-2 to a situation in which a plane sound wave may be represented in terms of the discrete frequencies of a Fourier series. For the illustrative example it is assumed that the variation of excess pressure with time might be represented by the triangular wave form shown in the diagram of section a) of Derivation Summary 12-3. Section b) of this derivation summary gives the Fourier series expression for the excess pressure variation with the amplitude of the first-order term used as the reference pressure amplitude. The first to r^{th} component excess pressure amplitude ratio is plotted in section b) as a series of spikes placed at corresponding values of the order r. The identity of the squared amplitude ratio with the sound intensity ratio for the r^{th} component is recognized by Eq. (3) of Derivation Summary 12-3 and a discrete line plot. For the reasons discussed in Derivation Summary 12-2, the heights of the lines in the figure of section b) are proportional to the sound power flow across a unit area of the plane sound wave.

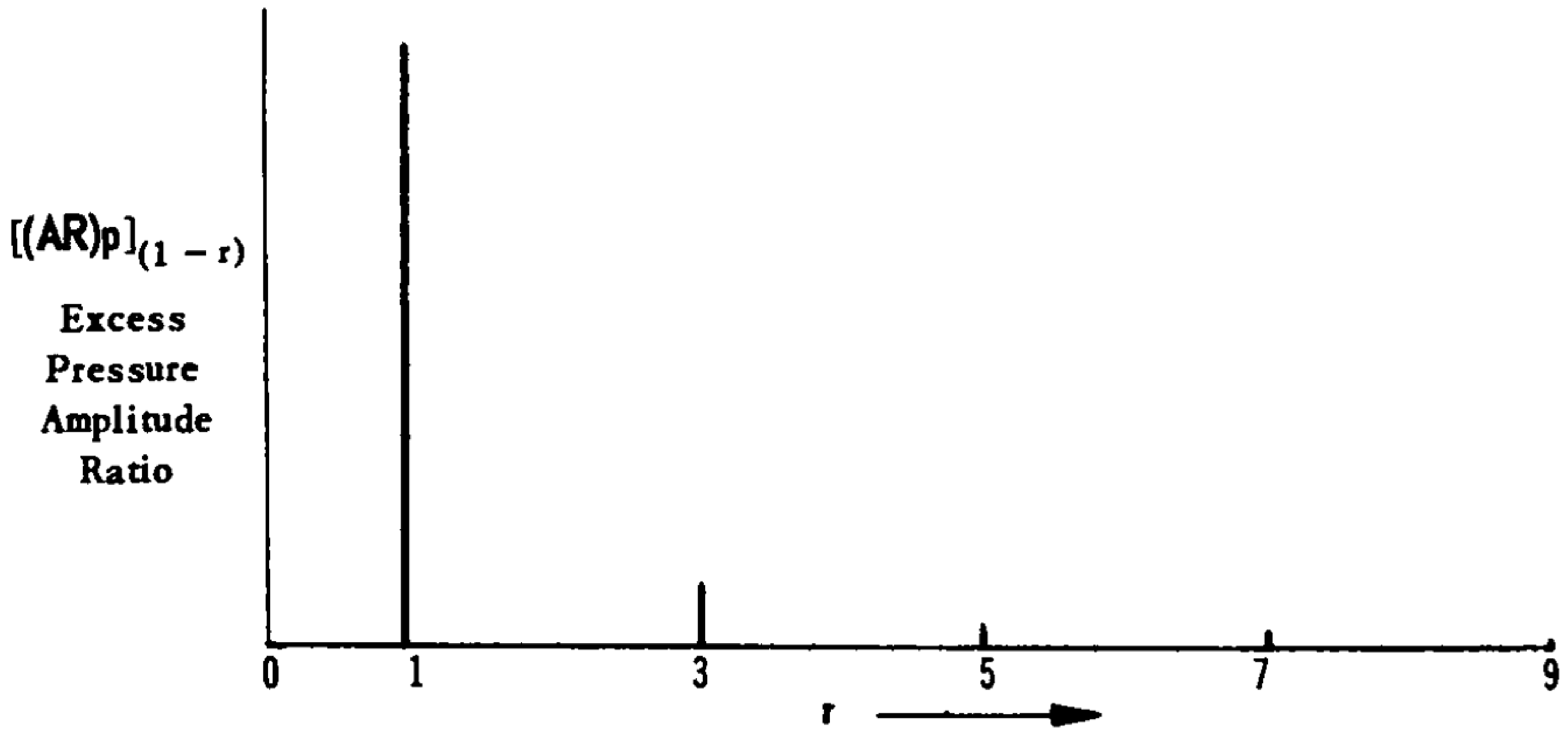
12.59. In practice, it is usually more convenient to use the total sound intensity as the reference quantity for forming intensity ratios than to use the intensity associated with any given component. This fact is recognized in the discussion of section d) of Derivation Summary 12-3, which gives a line plot of the total-to-r^{th} component sound intensity ratio corresponding to the line plot of section c). The relative heights of the spikes are the same in the plots of sections c) and d), but the ordinate scales must be different by the ratio of the first-order component intensity to the total intensity. Equation (8) of Derivation Summary 12-3 shows that the total-to-r^{th} component sound intensity ratio is equal to the squared reference to r^{th} component excess pressure amplitude ratio divided by the summation of all the squared amplitude ratios.

12.60. From the standpoint of physical meaning, the sound intensity ratio for the r^{th} component as defined in the last sentence gives the fraction of the total sound wave energy flow per unit area that is associated with the frequency of the r^{th} component. Equation (8) of Derivation Summary 12-3 is an expression for the sound intensity ratio in which the excess pressure amplitude ratio is defined in terms of a reference pressure instead of the pressure amplitude of the first

The relationships between frequency component amplitude ratios and plane wave sound intensity may be illustrated by a disturbance in which the instantaneous excess pressure variation has the isosceles triangular form shown below.



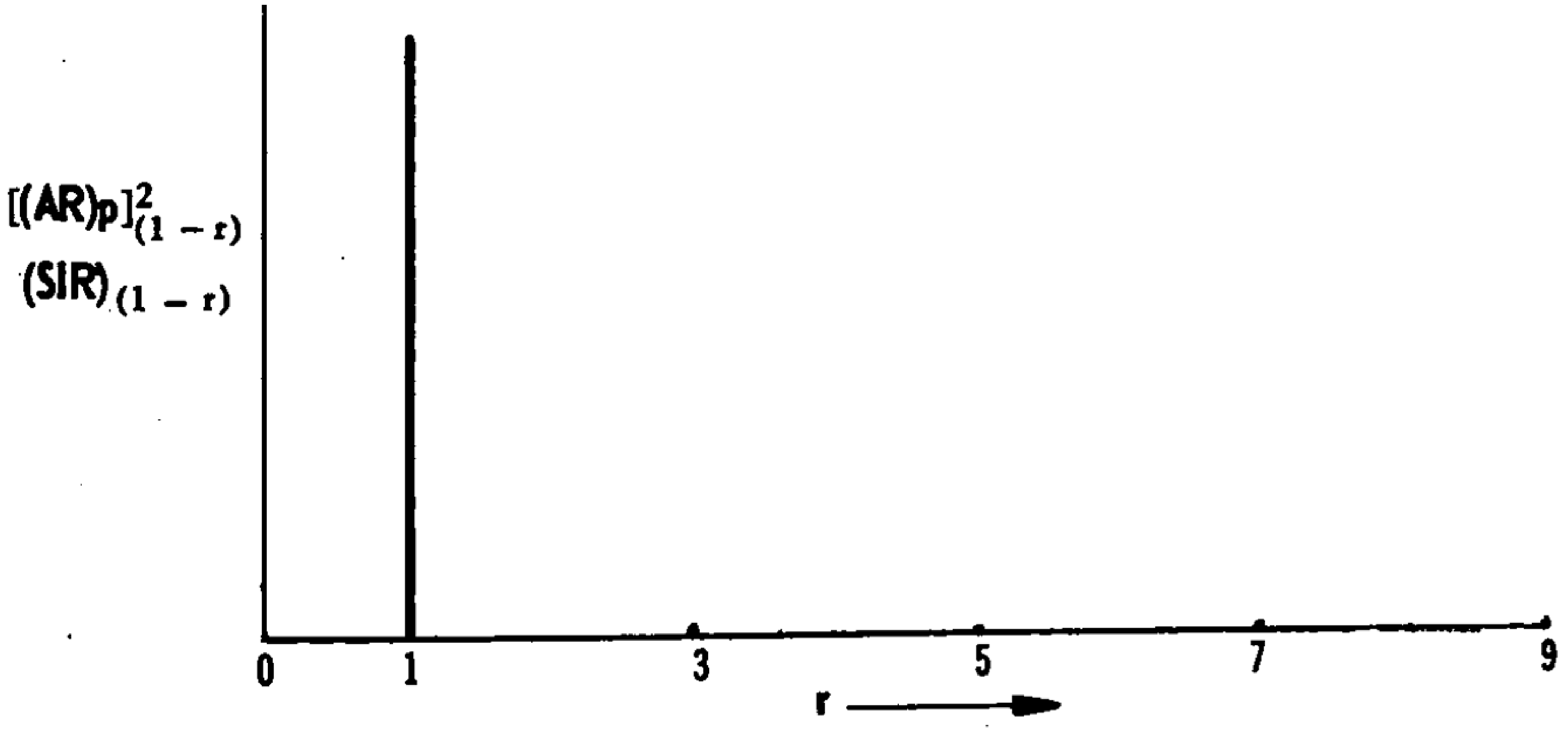
a) Pressure variation as a function of time for a triangular wave form



$$p(t) = p_{(1)a} \sum_{r=1}^{r=\infty} [(AR)p]_{(1-r)} \cos r w_1 t \qquad (1)$$

where $$[(AR)p]_{(1-r)} = \frac{1}{r^2} \quad \text{for } r = \text{odd integers} \qquad (2)$$

b) Excess pressure amplitude ratio distribution



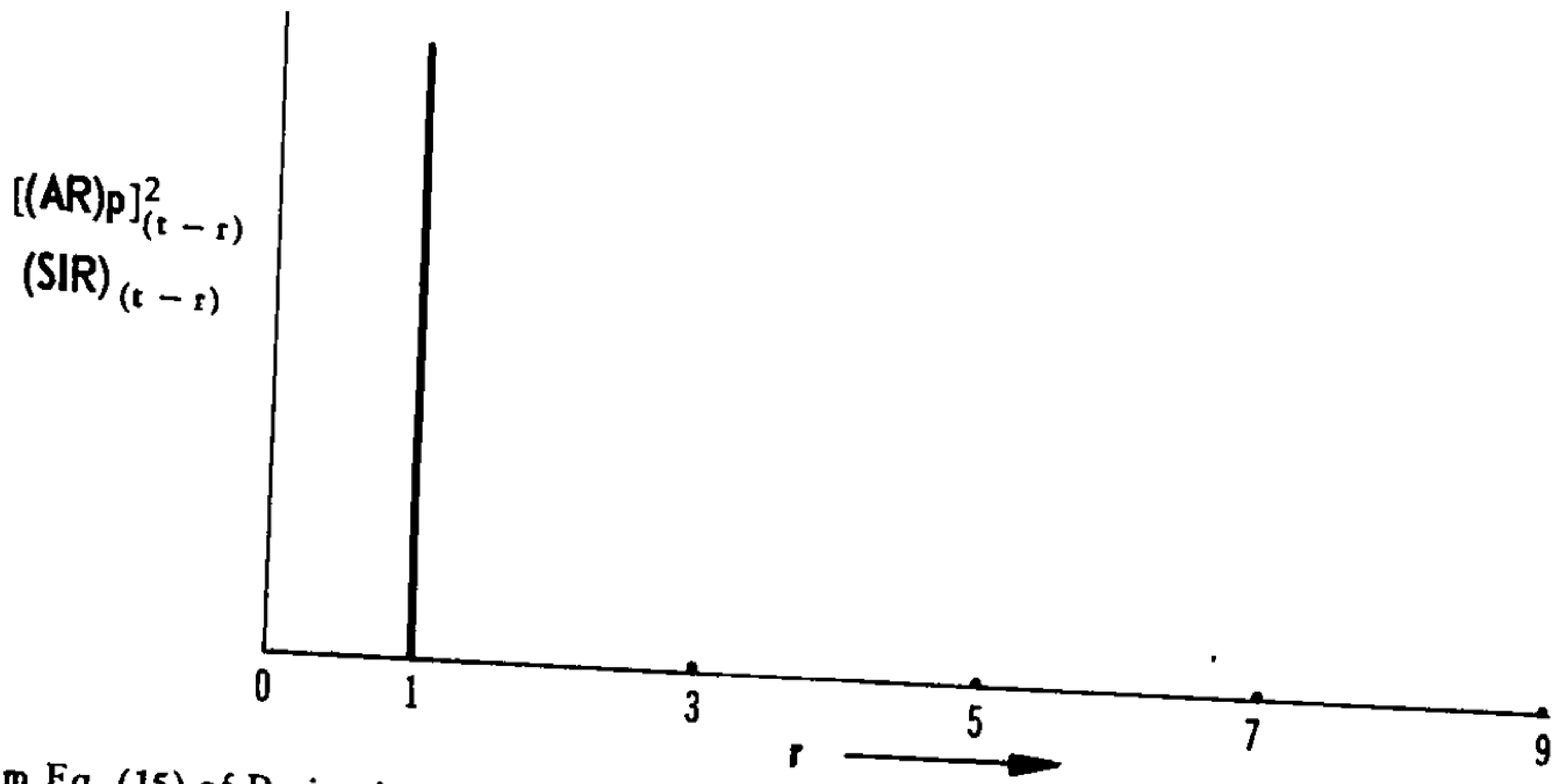
$$[(AR)p]^2_{(1-r)} = \frac{1}{r^4} \quad \text{for } r = \text{odd integers} \qquad (3)$$

From Eq. (12) of Derivation Summary 12-2, the squared first-to-rth component excess pressure ratio is equal to the first-to-rth component sound intensity ratio, i.e.

$$(SIR)_{(1-r)} = [(AR)p]^2_{(1-r)} \qquad (4)$$

c) First-to-rth component sound intensity ratio distribution

Derivation Summary 12-3. Amplitude ratio and sound intensity ratio distributions for a plane sound wave.
(Page 1 of 2)



From Eq. (15) of Derivation Summary 12-2, the total sound intensity is the summation of the sound intensity components associated with individual components.

The total-to-rth component sound intensity ratio is formed as the ratio of the intensity associated with the rth component to the total intensity, i.e.

$$(SIR)_{(t-r)} = \frac{(SI)_r}{(SI)_t} \tag{5}$$

where $(SI)_r$ = sound intensity associated with rth frequency component

$(SI)_t$ = total sound intensity

Using the expressions of Eqs. (8) and (15) of Derivation Summary 12-2

$$(SIR)_{(t-r)} = \frac{\dfrac{p^2_{(1)a}}{2\rho_o c}\,[(AR)p]^2_{(1-r)}}{\dfrac{p^2_{(1)a}}{2\rho_o c}\displaystyle\sum_{r=1}^{r=\infty}[(AR)p]^2_{(1-r)}} = \frac{[(AR)p]^2_{(1-r)}}{\displaystyle\sum_{r=1}^{r=\infty}[(AR)p]^2_{(1-r)}} \tag{6,7}$$

Also

$$(SIR)_{(t-r)} = \frac{[(AR)p]^2_{(ref-r)}}{\displaystyle\sum_{r=1}^{r=\infty}[(AR)p]^2_{(ref-r)}} \tag{8}$$

where

$[(AR)p]_{(ref-r)}$ = Reference pressure – rth excess pressure component amplitude ratio

$= \dfrac{p_{(r)a}}{p_{ref}}$

p_{ref} = any arbitrarily chosen constant pressure

d) Total-to-rth component sound intensity ratio

Derivation Summary 12-3. Amplitude ratio and sound intensity ratio distributions for a plane sound wave.
(Page 2 of 2)

component. Because this reference pressure appears symmetrically in the numerator and denominator terms, its magnitude cancels out without affecting the sound intensity ratio.

CONTINUOUS FREQUENCY POWER DISTRIBUTIONS; POWER SPECTRAL DENSITY

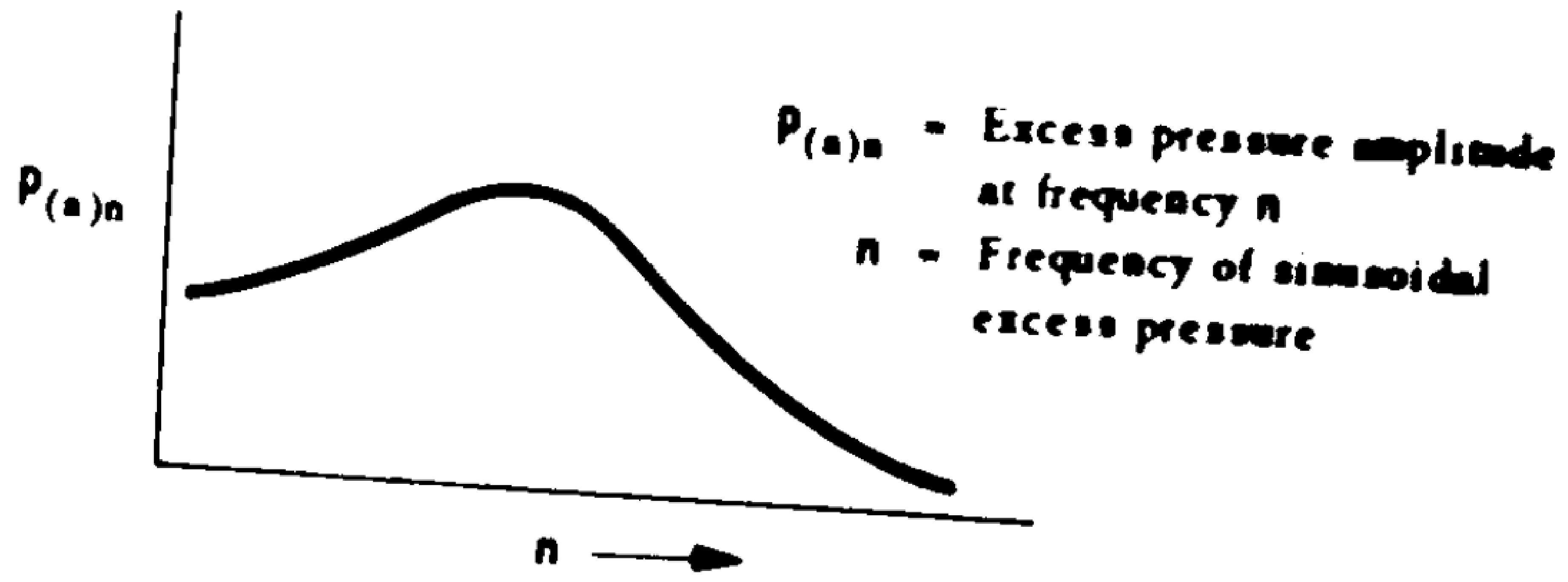
12.61. Sound waves may be made up of sinusoidal components with a continuous distribution of frequencies like that illustrated in section a) of Derivation Summary 12-4. The curve of this section is shown with excess pressure amplitude as a function of frequency as the ordinate, and frequency as the abscissa. The excess pressure distribution curve of section a) may be nondimensionalized by dividing the excess pressure amplitude by a reference pressure to produce the excess pressure amplitude ratio as a function of frequency ratio. The symbol for the excess pressure amplitude ratio is [(AR)(p)](FR), where the symbol (FR) shows that the amplitude ratio is considered as a function of frequency ratio. The excess pressure amplitude ratio distribution corresponding to the pressure amplitude distribution of section a) is shown in section b) of Derivation Summary 12-4.

12.62. By analogy to the relationships given in sections a), b), c), and d) of Derivation Summary 12-3, the squared amplitude ratio for a given frequency ratio may be considered as representing the plane sound wave power flow associated with an infinitesimally small range of frequency ratios centered about the given frequency ratio. This means that the differential increment in sound intensity considered as a function of frequency ratio is equal to a coefficient which includes the square of the excess pressure amplitude ratio, considered as a function of frequency ratio, multiplied by the differential frequency ratio band. The multiplier for the squared amplitude ratio in the coefficient is the square of the reference pressure divided by the product of the equilibrium density and the velocity of sound. The differential equation for sound intensity in terms of frequency ratio is given as Eq. (1) of Derivation Summary 12-4. Equation (2) of Derivation Summary 12-4 shows the total sound intensity as the integral of the squared excess pressure amplitude ratio over the zero-to-infinity range of frequency ratio multiplied by the coefficient of the amplitude ratio term in Eq. (1) of Derivation Summary 12-4.

SOUND INTENSITY DENSITY RATIO

12.63. Equation (3) of Derivation Summary 12-4 for the sound intensity — frequency ratio density appears when both sides of the basic relationship of Eq. (1) of Derivation Summary 12-4 are divided by the frequency ratio differential. This quantity gives the rate of change of sound intensity with respect to the frequency ratio; i.e., it describes the "density" of sound intensity per unit of the frequency ratio band. The total sound intensity is given by the integration of the sound

In practice, sound waves may be made up of pressure variations containing sinusoidal components with a continuous band of frequencies. An excess pressure amplitude distribution plot for a situation of this kind might have a shape like that illustrated below.



a) Illustrative excess pressure amplitude distribution plot for a continuous band of frequencies

Pressure amplitudes may be reduced to nondimensional terms by using a reference excess pressure as a common divisor, i.e.,

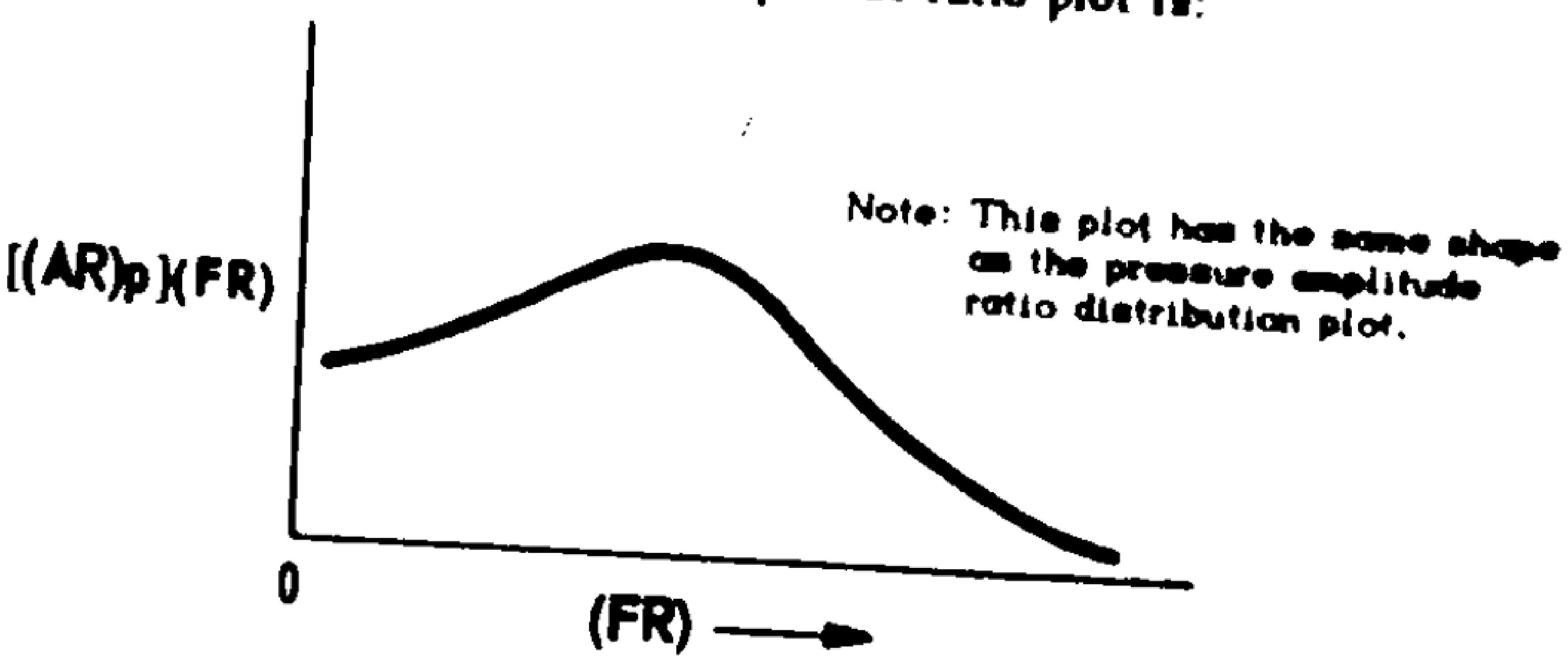
$$[(AR)p](FR) = \text{Excess pressure amplitude ratio as a function of frequency ratio} = \frac{p_a(FR)}{p_{ref}}$$

$p_a(FR)$ = Pressure amplitude as a function of frequency ratio

Frequency as the independent variable for distribution plots may be transformed to the nondimensional frequency ratio by using a reference frequency as the common divisor, i.e.,

$$(FR) = \frac{n}{n_{ref}} = \text{Frequency ratio} \quad ; \quad n_{ref} = \text{Reference frequency}$$

In terms of these variables, the excess pressure amplitude ratio plot is:



b) Illustrative excess pressure amplitude ratio distribution plot

Reasoning similar to that outlined in sections a), b), c) and d) of Derivation Summary 12-2 shows that the average sound power associated with a frequency ratio band $d(FR)$ of infinitesimal width may be represented by an expression of the form:

$$d\left[\frac{P_{av}}{(area)}\right] = d(SI)(FR) = \frac{p_{ref}^2}{2\rho_o c}\left[[(AR)p](FR)\right]^2 d(FR) \qquad (1)$$

Integration of this equation over the complete band of frequencies from 0 to ∞ gives the total power unit area of the plane wave associated with the sound, i.e.,

$$\frac{P_{av}}{(area)} = (SI)_t = \frac{p_{ref}^2}{2\rho_o c}\int_0^\infty \left[[(AR)p](FR)\right]^2 d(FR) \qquad (2)$$

where $(SI)_t$ = Total sound intensity.

Derivation Summary 12-4. Representation of physical variations by continuous frequency distributions: power density distributions. (Page 1 of 2)

Dividing both sides of Eq. (1) by d(FR) gives the sound intensity – frequency ratio density.

$$\frac{d(SI)(FR)}{d(FR)} = \frac{p_{ref}^2}{2\rho_o c}\left[[(AR)p](FR)\right]^2 \qquad (3)$$

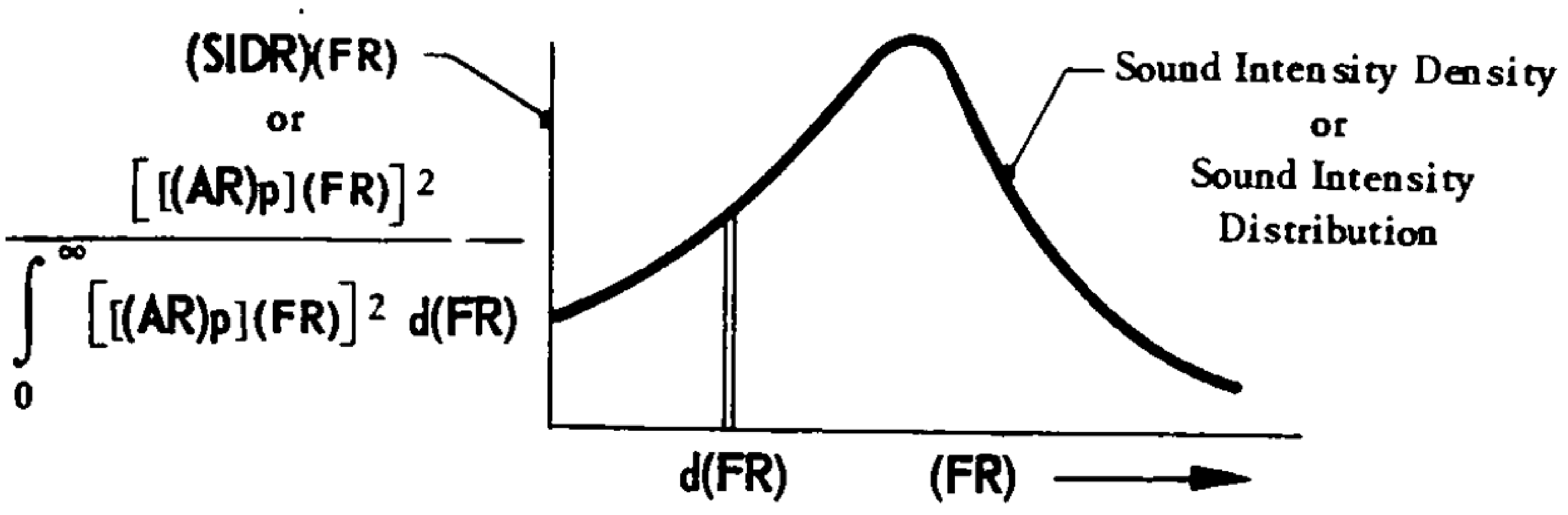
Dividing both sides of this equation by the total sound intensity gives the sound intensity ratio – frequency ratio density, or simply the sound intensity density.

$$(SIDR)(FR) = \frac{d\left[\frac{(SI)(FR)}{(SI)_t}\right]}{d(FR)} = \frac{\left[[(AR)p](FR)\right]^2}{\int_0^\infty \left[[(AR)p](FR)\right]^2 d(FR)} \qquad (4)$$

(SIDR)(FR) = Sound intensity density ratio or sound intensity density ratio distribution

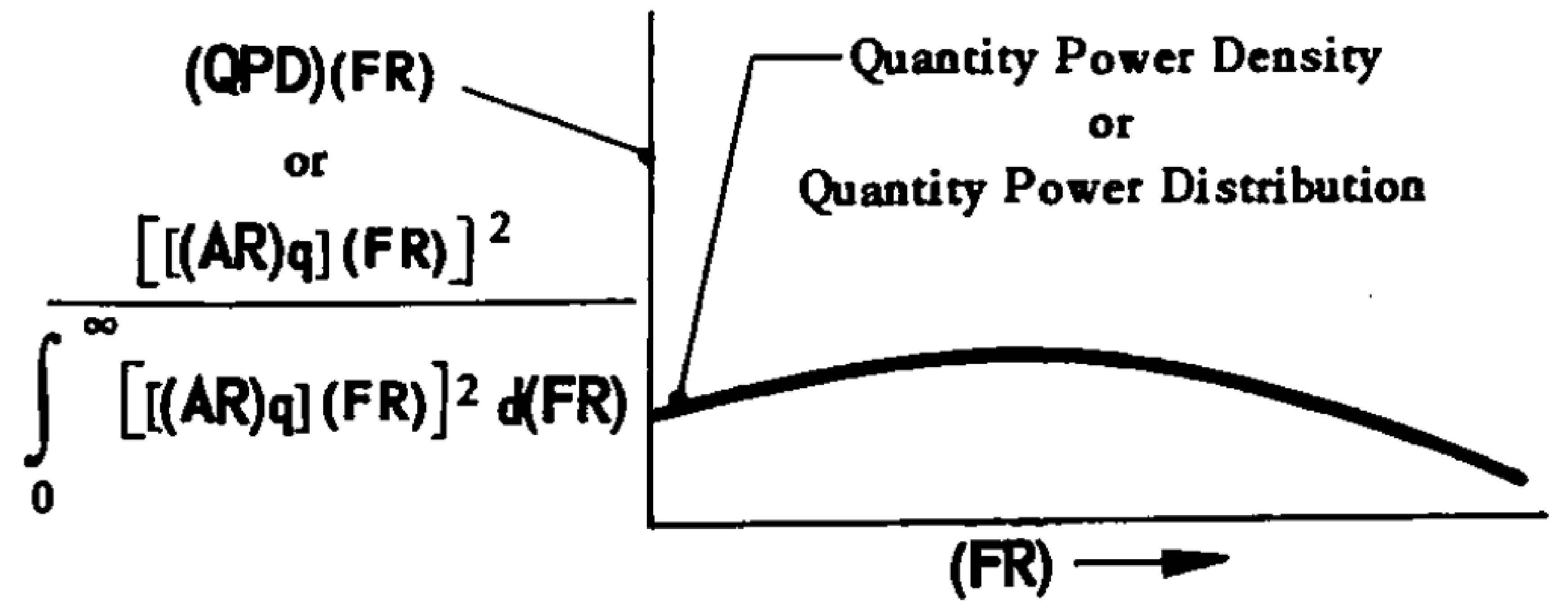
Graphically, the sound intensity density curve is determined by $\left[[(AR)p](FR)\right]^2$. The ordinate scale is determined by dividing $\left[[(AR)p](FR)\right]^2$ ordinates by the area under the curve expressed in terms of the integral

$$\int_0^\infty \left[[(AR)p](FR)\right]^2 d(FR)$$



c) Sound intensity distribution definitions and illustrative plot

The concepts, definitions and notation of the preceding sections are not restricted to the sound waves used as an illustrative example. Any physical variable that may be described by a continuous frequency distribution of sinusoidal components may be substituted for excess sound wave pressure. In symbols, this generalization is represented by substitution of q for p, and QP (to represent quantity power) for SI. In these generalized terms, the plot of section c) becomes:



(QPD)(FR) = Quantity power density as a function of frequency ratio or quantity power distribution

Note: Quantity power density as the term is defined here is analogous to the power spectral density (symbol PSD) as this term is used by Wiener (34) and Lee (25).

d) Quantity power density and power spectral density

Derivation Summary 12-4. Representation of physical variations by continuous frequency distributions; power density distributions. (Page 2 of 2) intensity density over the entire frequency ratio range. It is convenient to form the sound intensity density ratio as the ratio of the sound intensity density to the total sound intensity. This relationship is given by Eq. (4) of Derivation Summary 12-4.

12.64. The sound intensity density ratio may be represented as a function of frequency ratio in plots like that of section c) of Derivation Summary 12-4. Plots of this type are similar in shape to sound intensity plots, but have their ordinate scales modified to show the sound intensity density ratio as a function of frequency ratio. The modifying factor is the total sound intensity, which is proportional to the area under the sound intensity curve. Section c) of Derivation Summary 12-4 gives the relationships among the quantities in a plot of this type.

12.65. For engineering purposes, the sound intensity density ratio curve and a knowledge of the reference frequency and reference excess pressure are sufficient to describe a plane wave of sound.

GENERALIZATION OF CONCEPTS ASSOCIATED WITH PLANE WAVE SOUND INTENSITY; POWER SPECTRAL DENSITY

12.66. The concepts developed in Derivation Summaries 12-2, 12-3, and 12-4 for sound waves may be applied to situations that are describable by means of continuous frequency distributions, even though power as a simple physical quantity is not involved. All the symbols defined in Derivation Summaries 12-2, 12-3, and 12-4 may be applied to any physical situation if q (for physical quantity) is substituted for p, and the symbol (QPD) for quantity power density is used to replace the symbol (SIDR). Section d) of Derivation Summary 12-4 includes an illustrative plot of quantity power density as a function of frequency ratio and indicates the definition of quantity power density in terms of q as a generalized quantity.

12.67. Because the quantity power density plotted against frequency ratio corresponds to power density plotted against frequency for distribution of light intensity as a function of frequency in a conventional band spectrum, it corresponds to power spectral density as this term is defined by Wiener(34) and Lee.(25) These authors present rigorous treatments of the mathematical reasoning associated with power spectral density. Their works should be consulted by engineers who have occasion to make extensive use of the continuous frequency distributions method for describing physical situations.

CHAPTER 13

DESCRIPTION OF PERFORMANCE IN TERMS OF SINUSOIDAL RESPONSE CHARACTERISTICS

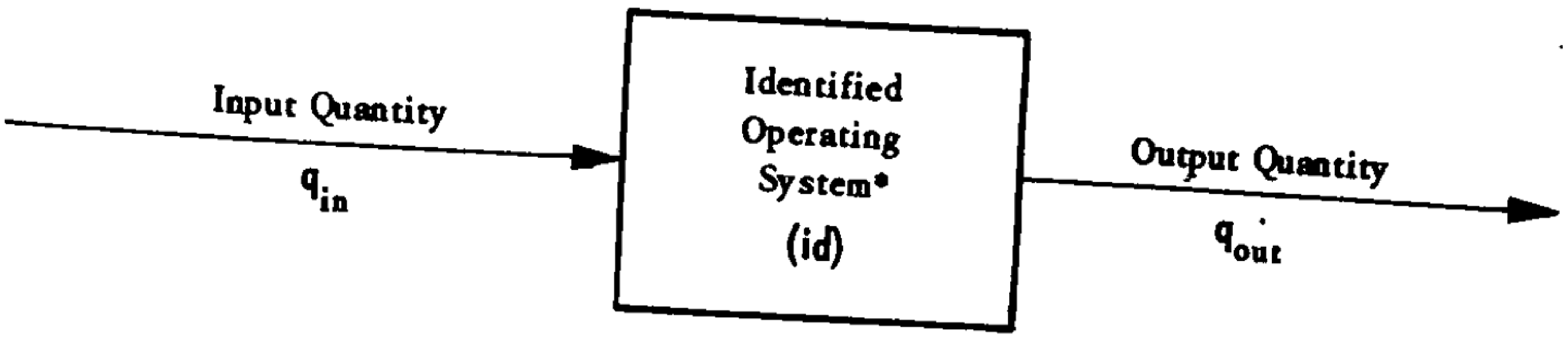
SINUSOIDAL PERFORMANCE CHARACTERISTICS

13.1. Sinusoidal and imaginary exponential forms appear as factors in the quantity-representing expressions described in Chapter 12. For this reason, it is important to define a system of concepts and symbols for representing sinusoidal forcing functions, sinusoidal performance characteristics of operating systems, and the corresponding output response variations. All the essential definitions may be made without discussing the procedural details required to find mathematical expressions for operating system response. Various methods for finding these expressions are derived in Volume II of this book. The discussion of this chapter is intended to build up the background for these derivations and to define a system of notation for describing situations that involve sinusoidal variations.

13.2. As a general rule, steady-state sinusoidal solutions are much simpler to handle than complete solutions which include both transient components and steady-state components as these components are defined in Chapter 6. For this reason, mathematical procedures to produce responses corresponding to sinusoidal forcing functions are usually restricted to finding solutions which apply only when the initial conditions are such that no transient component is present in the solution.

13.3. Mathematical reasoning to develop normal solutions for sinusoidal forcing functions is primarily based on finding the response of a single dependent variable corresponding to sinusoidal changes in a single independent variable. To meet the requirements for describing these normal solutions, a system is developed in this section to describe functional relationships between a single pair of variables. The concepts defined in this chapter for a single independent variable — dependent variable pair may be applied in the study of any situation, however complicated it may be in terms of variables or operating components.

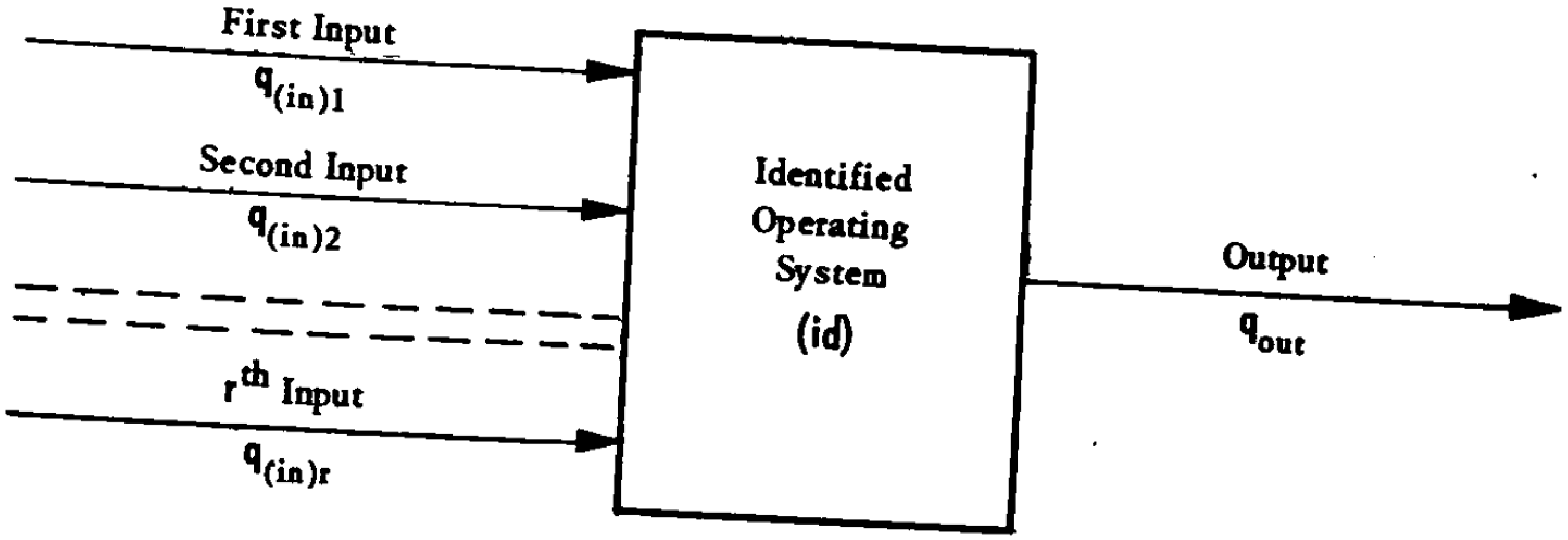
13.4. The typical situation associated with sinusoidal changes is illustrated by the functional diagram of section a) of Definition Summary 13-1, which describes the relationship between a single input and a single output established by an identified operating system in terms of the input-output performance function of the (id) identified operating system. In order to supply immediately available definitions of sinusoidal performance-describing concepts and the corresponding symbols, the notation is written in complete form for the definition summaries



$$q_{out} = [PF]_{(id)[q_{in};\, q_{out}]}\, q_{in} \tag{1}$$

* For the purposes of this diagram, the term Operating System is considered to include Operating Component as a special case.

a) Functional diagram and performance equation for operating components with a single input and a single output.



In general

$$q_{out} = [PF]_{(id)[q_{(in)1},\, q_{(in)2},\, \ldots,\, q_{(in)r};\, q_{out}]}(q_{(in)1}\, q_{(in)2} \ldots q_{(in)r}) \tag{2}$$

In many cases, the inputs may be combined into input function products, i.e.,

$$(IFP)_1 = (IFP)_{(1,2,\ldots,n)} = (IFP)_{[q_{(in)1},\, q_{(in)2},\, \ldots,\, q_{(in)n}]}$$
$$= f_1(q_{(in)1})f_2(q_{(in)2}) \ldots f_n(q_{(in)n})$$

$$(IFP)_2 = (IFP)_{(n+1,\, n+2,\, \ldots,\, r)} = (IFP)_{[q_{(in)(n+1)},\, q_{(in)(n+2)},\, \ldots,\, q_{(in)r}]}$$
$$= f_{(n+1)}(q_{(in)(n+1)})f_{(n+2)}(q_{(in)(n+2)}) \ldots f_r(q_{(in)r})$$

Using the input function form

$$q_{out} = [PF]_{(id)[(IFP)_1,\, (IFP)_2,\, \ldots;\, q_{out}]}(IFP)_1 (IFP)_2 \ldots \tag{3}$$

b) Functional diagram and performance equation forms for operating systems with multiple inputs and a single output

Definition Summary 13-1. Performance equations for multiple- and single-input operating systems. (Page 1 of 2)

In many situations it is convenient to use quantity transformations (see Definition Summary 10-3) for replacing the actual input or input function products by reference quantities; i.e., for a single input $q_{ref} \approx q_{in}$

(For generalized quantity transformation forms see Definition Summary 10-8.)

For multiple inputs

$$q_{(ref)1} \approx (\mathrm{IFP})_1 \qquad q_{(ref)2} \approx (\mathrm{IFP})_2$$

In terms of reference quantities, performance equations for multiple input operating systems have the form

$$q_{out} = [\mathrm{PF}]_{(id)[q_{(ref)1}, q_{(ref)2}, \ldots ; q_{out}]}(q_{(ref)1}, q_{(ref)2} \ldots) \tag{4}$$

When only a single reference quantity is involved, the performance equation reduces to the form

$$q_{out} = [\mathrm{PF}]_{(id)[q_{ref}; q_{out}]} q_{ref} \tag{5}$$

If a quantity transformation is used to replace the actual output quantity by a dependent quantity q_{dep}, Eq. (5) becomes

$$q_{dep} = [\mathrm{PF}]_{(id)[q_{ref}; q_{dep}]} q_{ref} \tag{6}$$

Equation (6) is the form of greatest usefulness for the development of mathematical solutions to describe operating system performance.

In working variable form

$$v = [\mathrm{RF}]u = (\mathrm{RAR})[\mathrm{RFRF}]u = (\mathrm{RAR})\ {}'[\mathrm{RF}]u \tag{7}$$

or

$$v = \eta u = \rho_r {}'\eta u \tag{8}$$

where

$u \approx q_{ref}$

$v \approx q_{dep}$

$[\mathrm{RF}] \approx \eta \approx [\mathrm{PF}]_{(id)[q_{ref}; q_{dep}]}$ = relating function for u to v

$\rho_r \approx (\mathrm{RAR})$ = reference amplitude ratio

${}'\eta \approx [\mathrm{RFRF}] = \dfrac{[\mathrm{RF}]}{(\mathrm{RAR})}$ = relating function ratio function

$= {}'[\mathrm{RF}]$ = nondimensional relating function

c) Performance function form for a single reference quantity and a single dependent quantity

Definition Summary 13-1. Performance equations for multiple- and single-input operating systems. (Page 2 of 2) of this section. This complete form notation is accompanied by the simple working variable symbols that are used in Volume II for the derivation of sinusoidal response forms.

PERFORMANCE REPRESENTATION FOR SYSTEMS WITH MULTIPLE INPUTS

13.5. A special situation of multiple inputs that may be treated in simplified terms is represented in section b) of Definition Summary 13-1. In this situation, an indefinite number of input quantities controls a single output quantity. Under practical conditions it is often possible to establish functions of the inputs that may be combined into one or more *input function products.* For example, the field current and the armature current of a direct-current motor are inputs that may be combined into a single input function product to describe the essential input condition for the motor. Each input function product may be considered as a *reference quantity* that combines with other similar reference quantities to determine the output quantity of an identified operating component.

13.6. Equation (6) of Definition Summary 13-1 gives the *single reference quantity — single dependent quantity performance function form* that will be used in this chapter to illustrate sinusoidal performance-describing concepts and symbols. The complete symbols are accompanied by symbols that may be used for working variable purposes. For example, u is suggested as the working symbol to represent the independent quantity, v is suggested for the dependent quantity, and the *relating function* (symbol [RF]) is shown as corresponding to the performance function. Because working variable forms are used for the purpose of mathematical derivations or the description of performance in situations so simple that no misunderstanding of symbols is possible, identifying subscripts are omitted from working variable symbols.

13.7. Equation (7) of Definition Summary 13-1 illustrates the concept of the *relating function ratio function* (main symbol [RFRF]), which is formed by dividing the relating function by an arbitrarily chosen *reference amplitude ratio* (main symbol (RAR)), with dimensions identical with those of the relating function. Because of this selection of dimensions, the relating function ratio function is always without dimensions, and for this reason it is also called the *nondimensional relating function* (symbol ′[RF]). The working variable symbol for the reference amplitude ratio is the small Greek letter rho (ρ) with r (for reference) as its subscript. The working variable symbol for the nondimensional relating function is the small Greek letter eta (η) with a preceding prime.*

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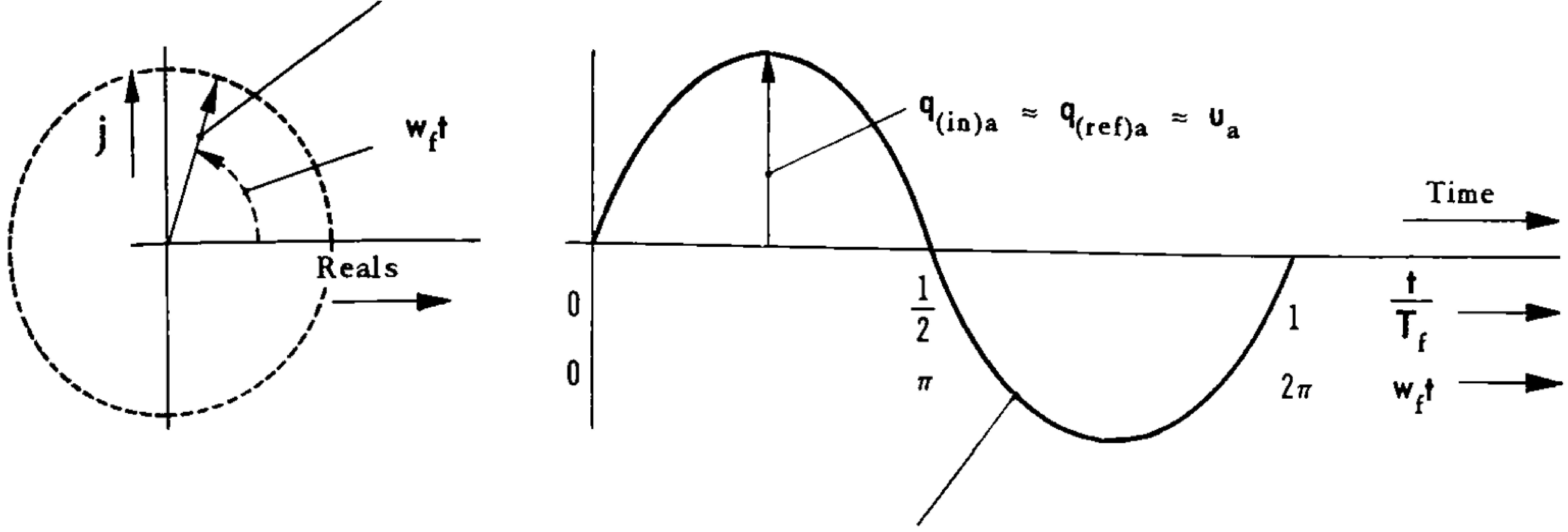
* The use of a preceding prime on the main (RF) symbol to show that the relating function has been reduced to nondimensional terms is an example of generalized convention in which the preceding prime indicates that the main symbol with which it is associated is a nondimensional form.

Sinusoidal input variations may be represented by either the rotating vector complex plane method (commonly called the clock diagram) or the sine plot method of Fig. 12-3.

$$q_{in} = q_{(in)a} e^{jw_f t} \approx q_{ref} = q_{(ref)a} e^{jw_f t} \approx u = u_a e^{jw_f t} \qquad (1)$$



$$q_{in} = q_{(in)a} \sin w_f t \approx q_{ref} = q_{(ref)a} \sin w_f t \approx u = u_a \sin w_f t \qquad (2)$$

q_{ref} = reference forcing function

q_{in} = input forcing function

u = working variable forcing function

$q_{(in)a}$ = input quantity amplitude

$q_{(ref)a}$ = reference quantity amplitude

u_a = working variable amplitude

w_f = angular forcing frequency $= 2\pi n_f = \frac{2\pi}{T_f}$

n_f = forcing frequency $= \frac{1}{T_f}$

$T_f = \frac{1}{n_f}$ = forcing period

a) Representation of sinusoidal forcing functions.

In general, the action of an operating component is to establish a functional relationship between an input, q_{in}, and a corresponding output, q_{out}. This action is represented by the input-output performance function for the identified operating component, i.e.,

$$q_{out} = [PF]_{(id)[q_{in};\, q_{out}]}\, q_{in} \approx q_{dep} = [PF]_{(id)[q_{ref};\, q_{dep}]}\, q_{ref} \qquad (3,4)$$

and in working variable form

$$q_{out} \approx v = \eta u = \rho_r{}' \eta u \qquad (5)$$

b) Generalized forms for representing the performance of operating components

Definition Summary 13-2. Representation of sinusoidal response characteristics for operating systems.
(Page 1 of 4)

The performance of an operating component may be described in terms of sinusoidal performance characteristics. The action of any operating component whose sinusoidal performance characteristics depend only on the frequency of pure sinusoidal input variations is to produce a sinusoidal output with the same frequency as a simple sinusoidal input. The output is shifted from the input by a dynamic response angle (DRA) and the ratio of the output amplitude to the input amplitude is the input-output dynamic amplitude ratio (main symbol (DAR)). In general, the dynamic response angle and the dynamic amplitude ratio are both functions of the forcing frequency n_f. In formal terms

$$q_{out} \approx q_{dep} = [\mathbf{PF}]_{(id)[q_{ref};q_{dep}]sin}\, q_{(ref)a}\, e^{jw_f t} \tag{6}$$

$$q_{out} = [\mathbf{PF}]_{(id)[q_{in};q_{out}]sin}\, q_{(in)a}\, e^{jw_f t} \tag{7}$$

where

$$q_{out} \approx \upsilon = \eta \upsilon_a e^{jw_f t} = {'\eta} \upsilon_a e^{jw_f t} \tag{8}$$

$[\mathbf{PF}]_{(id)[q_{in};q_{out}]sin}$ = input – output sinusoidal performance function for the (id) identified operating component

$[\mathbf{PF}]_{(id)[q_{ref};q_{dep}]sin}$ = reference quantity – dependent quantity sinusoidal performance function for the (id) identified operating component

η = working variable relating function

ρ_r = working variable reference amplitude ratio

In general, it is possible to express the performance function in terms of the product of the dynamic amplitude ratio and an imaginary exponential with the dynamic response angle in its exponent.

$$[\mathbf{PF}]_{(id)[q_{in};\, q_{out}]sin} = (\mathbf{DAR})_{(id)[q_{in};\, q_{out}]}\, e^{j(DRA)_{(id)[q_{in};q_{out}]}} \tag{9}$$

$$[\mathbf{PF}]_{(id)[q_{ref};\, q_{dep}]sin} = (\mathbf{DAR})_{(id)[q_{ref};\, q_{dep}]}\, e^{j(DRA)_{(id)[q_{ref};q_{dep}]}} \tag{10}$$

$$\eta = \rho e^{j\phi} = \rho_r {'\eta} = \rho_r \frac{\rho}{\rho_r} e^{j\phi} \tag{11}$$

$(\mathbf{DAR})_{(id)[q_{in};q_{out}]}$ = input – output dynamic amplitude ratio for the (id) identified operating component

$(\mathbf{DAR})_{(id)[q_{ref};q_{dep}]}$ = reference variable – dependent variable dynamic amplitude ratio for the (id) identified operating component

ρ = working variable dynamic amplitude ratio

$(\mathbf{DRA})_{(id)[q_{in};q_{out}]}$ = input – output dynamic response angle

$(\mathbf{DRA})_{(id)[q_{ref};q_{dep}]}$ = reference quantity – dependent quantity dynamic response angle

ϕ = working variable dynamic response angle

Sinusoidal response characteristics may be represented by vectors drawn on the complex plane.

Definition Summary 13-2. Representation of sinusoidal response characteristics for operating systems.
(Page 2 of 4)

In general, dynamic amplitude ratio and dynamic response angle may be expressed as functions of forcing frequency, n_f, forcing period, T_f, angular forcing frequency, w_f, and forcing frequency ratio (shortened to frequency ratio when no misunderstanding is possible) i.e.,

$$(\mathrm{DAR})(n_f) \approx (\mathrm{DAR})(T_f) \approx (\mathrm{DAR})(w_f) \approx (\mathrm{DAR})(\mathrm{FFR}) \approx (\mathrm{DAR})(\mathrm{FR}) \approx (\mathrm{DAR})(\beta) \approx \rho(\beta) \tag{12}$$

$$(\mathrm{DRA})(n_f) \approx (\mathrm{DRA})(T_f) \approx (\mathrm{DRA})(w_f) \approx (\mathrm{DRA})(\mathrm{FFR}) \approx (\mathrm{DRA})(\mathrm{FR}) \approx (\mathrm{DRA})(\beta) \approx \phi(\beta) \tag{13}$$

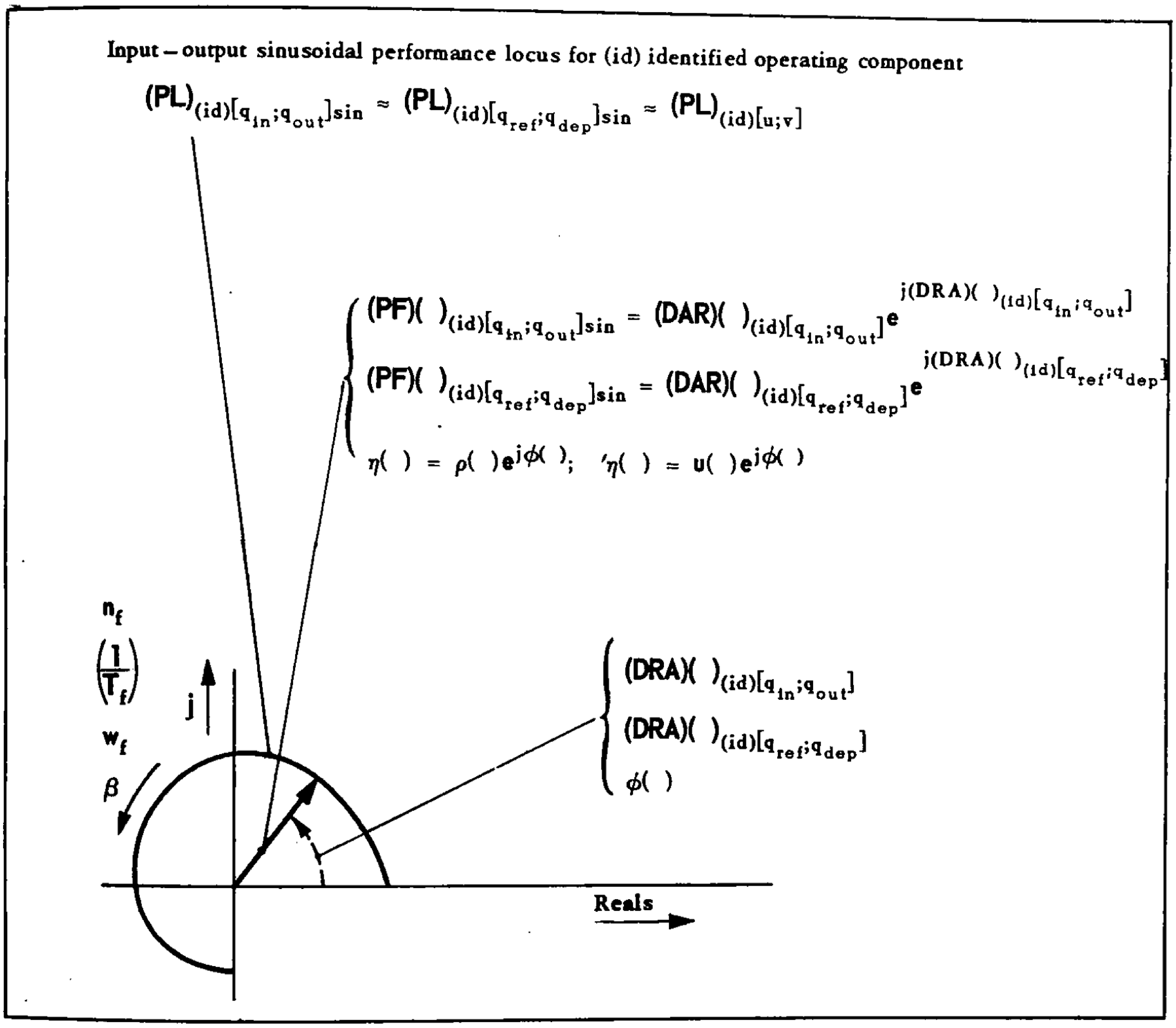
where

$$(\mathrm{FFR}) = \frac{n_f}{n_{ref}} = \text{forcing frequency ratio}$$

$\approx (\mathrm{FR})$ = frequency ratio

$\approx \beta$ = forcing frequency ratio or frequency ratio

n_{ref} = reference frequency (arbitrarily selected)


Input – output sinusoidal performance locus for (id) identified operating component
$(\mathrm{PL})_{(id)[q_{in};q_{out}]\sin} \approx (\mathrm{PL})_{(id)[q_{ref};q_{dep}]\sin} \approx (\mathrm{PL})_{(id)[u;v]}$
$(\mathrm{PF})(\ )_{(id)[q_{in};q_{out}]\sin} = (\mathrm{DAR})(\ )_{(id)[q_{in};q_{out}]}e^{j(\mathrm{DRA})(\ )_{(id)[q_{in};q_{out}]}}$
$(\mathrm{PF})(\ )_{(id)[q_{ref};q_{dep}]\sin} = (\mathrm{DAR})(\ )_{(id)[q_{ref};q_{dep}]}e^{j(\mathrm{DRA})(\ )_{(id)[q_{ref};q_{dep}]}}$
$\eta(\ ) = \rho(\ )e^{j\phi(\ )};\quad {}'\eta(\ ) = u(\ )e^{j\phi(\ )}$
n_f
$\left(\frac{1}{T_f}\right)$
w_f
β
j
$(\mathrm{DRA})(\ )_{(id)[q_{in};q_{out}]}$
$(\mathrm{DRA})(\ )_{(id)[q_{ref};q_{dep}]}$
$\phi(\ )$
Reals

c) Representation of sinusoidal response characteristics

Definition Summary 13-2. Representation of sinusoidal response characteristics for operating systems. (Page 3 of 4)

In practice it is advantageous to express amplitude ratios in nondimensional forms by defining

$[(\mathrm{DAR})(\mathrm{RAR})\mathrm{R}]_{(id)[q_{in};q_{out}]}$ = input – output dynamic amplitude ratio – reference amplitude ratio for (id) identified operating component

$$= \frac{(\mathrm{DAR})_{(id)[q_{in};q_{out}]}}{(\mathrm{RAR})} = \frac{(\mathrm{AR})_{dyn}}{(\mathrm{AR})_{ref}} \approx \mu = \frac{\rho}{\rho_r}$$

= working variable dynamic amplitude ratio, or working variable nondimensional amplitude ratio

The input – output forcing function ratio equations appear when the output is divided by the input multiplied by the reference amplitude ratio. The nondimensional sinusoidal performance function in amplitude ratio – phase angle form is given by the right-hand side of the equation.

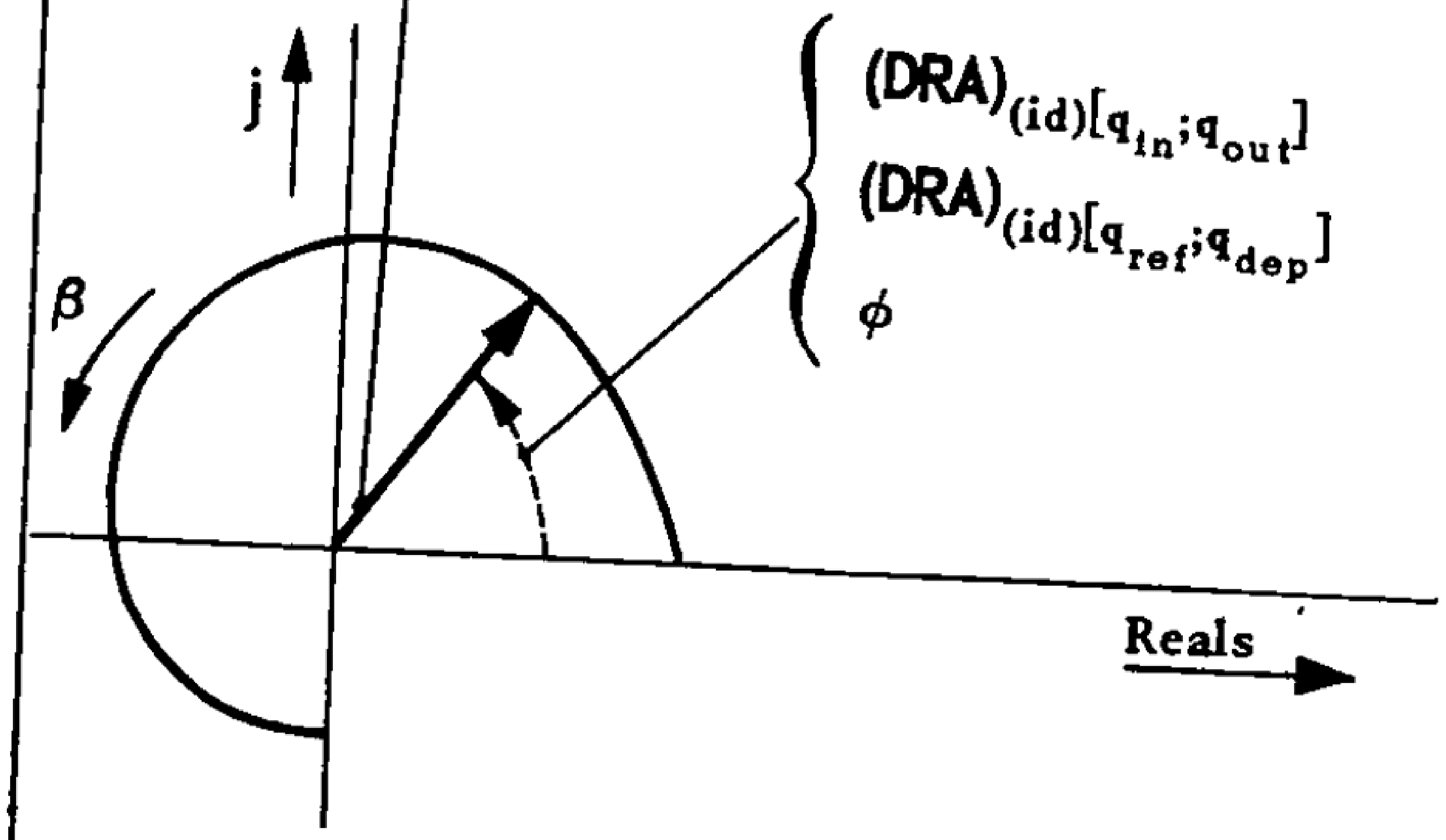
$$\frac{q_{out}}{(\mathrm{RAR})_{(id)[q_{in};q_{out}]}\,q_{in}} = [(\mathrm{DAR})(\mathrm{RAR})\mathrm{R}]_{(id)[q_{in};q_{out}]}\,e^{j(\mathrm{DRA})_{(id)[q_{in};q_{out}]}}$$

$$= [\mathrm{RFRF}]_{(id)[q_{in};q_{out}]} = {}'[\mathrm{RF}]_{(id)[q_{in};q_{out}]}$$

$$\frac{q_{dep}}{(\mathrm{RAR})_{(id)[q_{ref};q_{dep}]}\,q_{ref}} = [(\mathrm{DAR})(\mathrm{RAR})\mathrm{R}]_{(id)[q_{ref};q_{dep}]}\,e^{j(\mathrm{DRA})_{(id)[q_{ref};q_{dep}]}}$$

$$= [\mathrm{RFRF}]_{(id)[q_{ref};q_{dep}]} = {}'[\mathrm{RF}]_{(id)[q_{ref};q_{dep}]}$$

$$\frac{v}{\rho_r u} = {}'\eta = u e^{j\phi}$$


j
β
$(\mathrm{DRA})_{(id)[q_{in};q_{out}]}$
$(\mathrm{DRA})_{(id)[q_{ref};q_{dep}]}$
ϕ
Reals

The nondimensional sinusoidal performance function may be described by a complex plane vector representing the dimensionless relating function. The continuous curve passing through the end points of these vectors is the nondimensional performance locus for the (id) identified operating component.

d) Representation of sinusoidal response in nondimensional terms

Definition Summary 13-2. Representation of sinusoidal response characteristics for operating systems. (Page 4 of 4)

13.8. Section a) of Definition Summary 13-2 summarizes the concepts, diagrams, and symbols for describing a simple sinusoidal forcing function. A forcing function of this type may be represented by an imaginary exponential and a vector rotating about the origin of the complex plane, or by a plot of a sine function with $w_f t$ as its argument and the amplitude of the variation as its coefficient.

13.9. To show that the same forms and diagrams may be applied to various situations, input and output symbols, reference quantity – dependent quantity symbols, and working variable symbols are included in section a) of Definition Summary 13-2. Section b) gives the generalized performance function forms for describing the sinusoidal performance characteristics of an operating component. It is desirable to restrict generalized considerations to forms that depend only on the forcing frequency. The forms that fall into this class are particularly useful because they satisfactorily represent many practical situations and at the same time are convenient for mathematical reasoning. All the sinusoidal performance-describing concepts and symbols of Definition Summary 13-2 and later summaries are subject to this restriction.

13.10. Section c) of Definition Summary 13-2 shows that the sinusoidal response characteristics of an operating component may be represented by a performance locus plotted on the complex plane. This performance locus is a plot of the end of the performance function vector, which has a length proportional to the dynamic amplitude ratio and is located at an angle equal to the dynamic response angle from the positive direction of the axis of reals. Each point on the performance locus corresponds to a frequency of the forcing function or some quantity proportional to this frequency. For example, the reciprocal of the forcing period, T_f, is identical with the forcing frequency, and the angular forcing frequency is equal to 2π times the forcing frequency. The forcing frequency ratio and its equivalent, the frequency ratio, which is defined as the ratio of the forcing frequency to an arbitrarily selected reference frequency, may also be used to determine points on the performance locus.

13.11. Section d) of Definition Summary 13-2 shows the representation of sinusoidal performance characteristics in nondimensional terms. The nondimensional performance function expressions are formed from the performance expressions of section c) when the dynamic amplitude ratio is divided by an arbitrarily selected reference amplitude ratio. In any given situation, the reference amplitude ratio is chosen so that the values of the dynamic amplitude ratio – reference amplitude ratio ratio, which is the nondimensional sinusoidal performance function amplitude, and the nondimensional relating function change over a convenient range of levels. For example, the reference amplitude ratio is often made equal to the dynamic amplitude ratio over the operating region of greatest interest so that the important values of the dynamic amplitude ratio – reference amplitude ratio ratio are all near unity.

13.12. In working variable terms, the dynamic amplitude ratio – reference amplitude ratio ratio is given the lower case Greek letter mu (μ) as its symbol, and the corresponding dynamic response angle is given the lower case Greek letter phi (ϕ) as its symbol. The sinusoidal characteristic curves of Volume II, which give quantitative information on the nondimensional performance functions corresponding to various performance equation forms, are plotted with μ and ϕ as the dependent variables and the frequency ratio represented by β as the independent variable. The complex plane polar plot curves of Volume II are graphical representations of the dimensionless relating function.

REPRESENTATION OF SINUSOIDAL RESPONSE

13.13. Operating systems with linear characteristics* have the property of responding to a sinusoidal steady-state input forcing function with a steady-state sinusoidal output variation with a frequency equal to the forcing frequency. The equivalent angular argument for any given point of the response curve differs from the corresponding point of the input curve by the dynamic response angle. The response amplitude is equal to the input amplitude multiplied by the amplitude ratio. These facts are proved and the mathematical forms corresponding to a number of important cases are derived in Volume II of this book. Section a) of Definition Summary 13-3 lists the sinusoidal response expressions for a linear operating system. The most widely useful forms are given by Eqs. (6), (7), and (8), which are nondimensionalized when both sides are divided by the product of the input forcing function and a reference amplitude ratio. This product is the output forcing function, which forms a nondimensional ratio with the output.

13.14. Section b) of Definition Summary 13-3 shows the representation of the response to a steady-state sinusoidal forcing function in terms of the complex plane rotating vector diagram and also in terms of the sine wave diagram described in Fig. 12-3. On the complex plane diagram, the input forcing function is represented by a vector of length proportional to the input amplitude and rotating with the forcing angular velocity. The output response is represented by a vector rotating with the same angular velocity and displaced from the input forcing function vector by the dynamic response angle. The length of the response vector is equal to the length of the input forcing function vector multiplied by the dynamic amplitude ratio. The sine wave representation of the complex plane representation is shown on the right-hand side of section b) of Definition Summary 13-3.

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* The mathematical properties of linear systems are discussed in Chapter I of Transients in Linear Systems, by Gardner and Barnes (18).

The performance function forms of Definition Summary 13-2 operating on the sinusoidal input forcing function forms of Definition Summary 13-1 shows that

$$q_{out} = (PF)_{(id)[q_{in};\, q_{out}]} \sin q_{in} \tag{1}$$

$$= (PF)_{(id)[q_{in};\, q_{out}]} \sin q_{(in)a} e^{jw_f t} \tag{2}$$

$$= (DAR)_{(id)[q_{in};\, q_{out}]} e^{j(DRA)_{(id)(q_{in};q_{out})}} q_{(in)a} e^{jw_f t} \tag{3}$$

$$= (DAR)_{(id)[q_{in};\, q_{out}]} q_{(in)a} e^{j[w_f t + (DRA)_{(id)(q_{in};q_{out})}]} \tag{4}$$

$$= (RAR)_{(id)[q_{in};\, q_{out}]} [(DAR)(RAR)R]_{(id)(q_{in};\, q_{out})} q_{(in)a} e^{j[w_f t + (DRA)_{(id)(q_{in};q_{out})}]} \tag{5}$$

or, if the input multiplied by the reference amplitude ratio is taken as $q_{(out)f}$, output forcing function

$$\frac{q_{out}}{q_{(out)f}} = \frac{q_{out}}{(RAR)_{(id)[q_{in};\, q_{out}]} q_{in}} = [(DAR)(RAR)R]_{(id)(q_{in};\, q_{out})} e^{j(DRA)_{(id)(q_{in};q_{out})}} \tag{6}$$

The corresponding form when the input and output are replaced by the reference variable is

$$\frac{q_{dep}}{q_{(dep)f}} = \frac{q_{dep}}{(RAR)_{(id)[q_{ref};\, q_{dep}]} q_{ref}} = [(DAR)(RAR)R]_{(id)(q_{ref};\, q_{dep})} e^{j(DRA)_{(id)(q_{ref};q_{dep})}} \tag{7}$$

when

$U \approx q_{in}$ = working variable form of independent variable

$V \approx q_{out}$ = working variable form of dependent variable

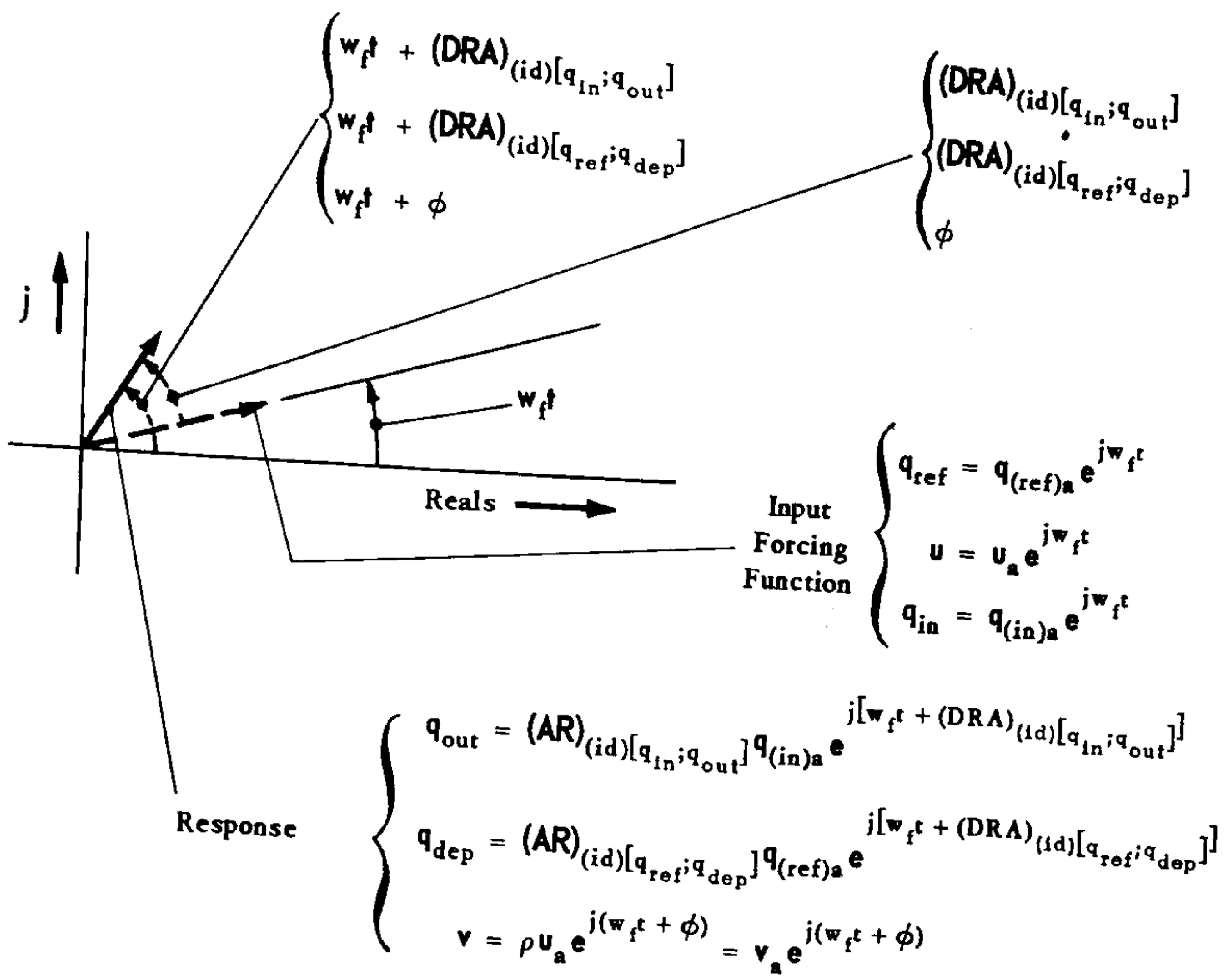
then

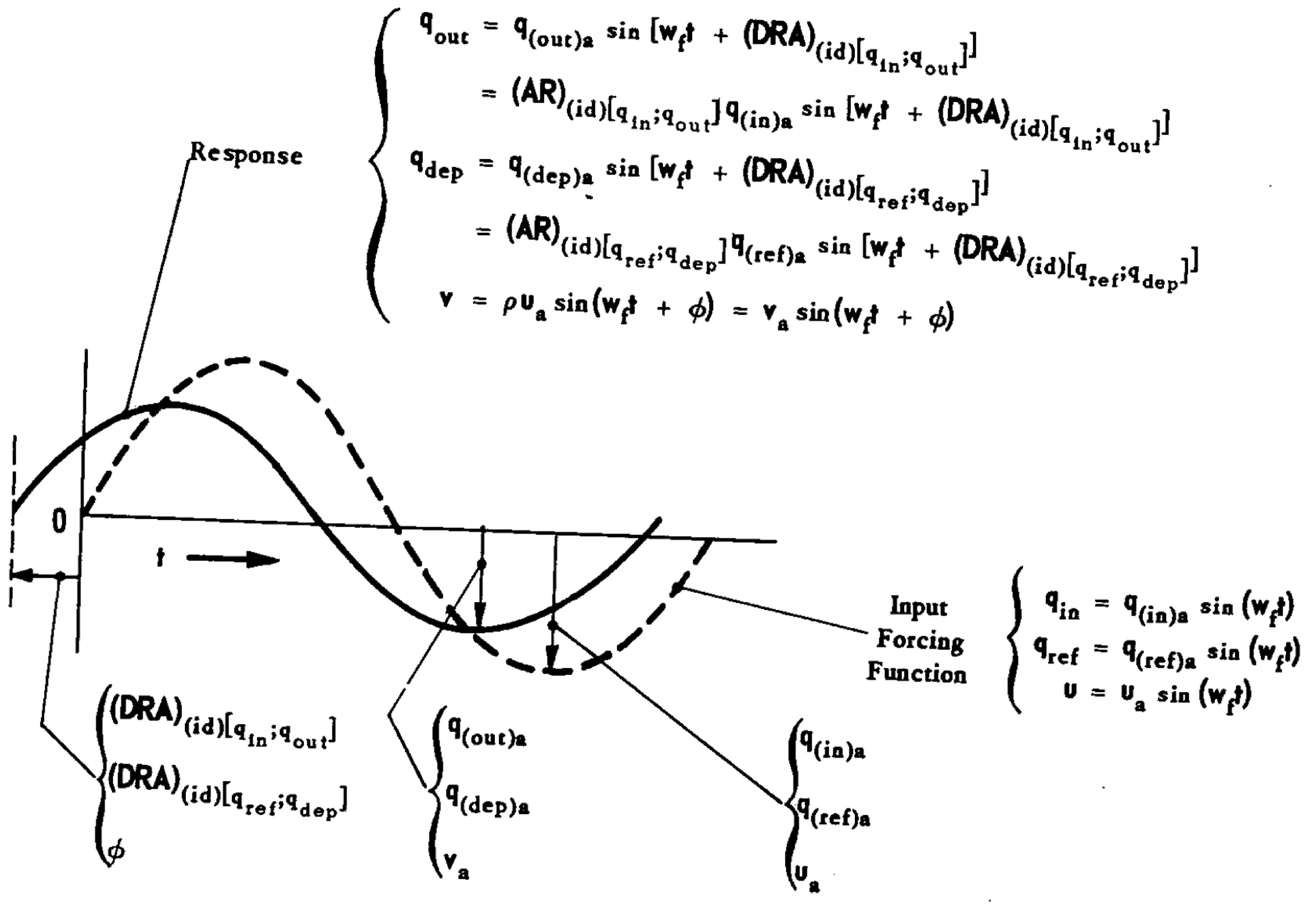
$\rho_r U = V_f$ = working variable form of dependent variable forcing function

$$\frac{V}{V_f} = \frac{V}{\rho_r U} = \mu e^{j\phi} \equiv \text{nondimensional sinusoidal response characteristic} \tag{8}$$

a) Sinusoidal response equations

Definition Summary 13-3. Representation of steady-state sinusoidal response for linear systems. (Page 1 of 3)


$\begin{cases} w_f t + (DRA)_{(id)[q_{in};q_{out}]} \\ w_f t + (DRA)_{(id)[q_{ref};q_{dep}]} \\ w_f t + \phi \end{cases}$
$\begin{cases} (DRA)_{(id)[q_{in};q_{out}]} \\ (DRA)_{(id)[q_{ref};q_{dep}]} \\ \phi \end{cases}$
j
$w_f t$
Reals
Input Forcing Function
$\begin{cases} q_{ref} = q_{(ref)a} e^{j w_f t} \\ u = u_a e^{j w_f t} \\ q_{in} = q_{(in)a} e^{j w_f t} \end{cases}$
Response
$\begin{cases} q_{out} = (AR)_{(id)[q_{in};q_{out}]} q_{(in)a} e^{j[w_f t + (DRA)_{(id)[q_{in};q_{out}]}]} \\ q_{dep} = (AR)_{(id)[q_{ref};q_{dep}]} q_{(ref)a} e^{j[w_f t + (DRA)_{(id)[q_{ref};q_{dep}]}]} \\ v = \rho u_a e^{j(w_f t + \phi)} = v_a e^{j(w_f t + \phi)} \end{cases}$


Response
$\begin{cases} q_{out} = q_{(out)a} \sin [w_f t + (DRA)_{(id)[q_{in};q_{out}]}] \\ \quad = (AR)_{(id)[q_{in};q_{out}]} q_{(in)a} \sin [w_f t + (DRA)_{(id)[q_{in};q_{out}]}] \\ q_{dep} = q_{(dep)a} \sin [w_f t + (DRA)_{(id)[q_{ref};q_{dep}]}] \\ \quad = (AR)_{(id)[q_{ref};q_{dep}]} q_{(ref)a} \sin [w_f t + (DRA)_{(id)[q_{ref};q_{dep}]}] \\ v = \rho u_a \sin(w_f t + \phi) = v_a \sin(w_f t + \phi) \end{cases}$
0
t
Input Forcing Function
$\begin{cases} q_{in} = q_{(in)a} \sin (w_f t) \\ q_{ref} = q_{(ref)a} \sin (w_f t) \\ u = u_a \sin (w_f t) \end{cases}$
$\begin{cases} (DRA)_{(id)[q_{in};q_{out}]} \\ (DRA)_{(id)[q_{ref};q_{dep}]} \\ \phi \end{cases}$
$\begin{cases} q_{(out)a} \\ q_{(dep)a} \\ v_a \end{cases}$
$\begin{cases} q_{(in)a} \\ q_{(ref)a} \\ u_a \end{cases}$

b) Representation of the steady-state response to a sinusoidal forcing function

Definition Summary 13-3. Representation of steady-state sinusoidal response for linear systems. (Page 2 of 3)

When the output forcing function for the identified operating component is defined as

$$(\mathrm{RAR})_{(id)[q_{in};\,q_{out}]}\,q_{in} = (\mathrm{OFF})_{(id)} = q_{(out)(f)a} \tag{9}$$

$$\frac{q_{out}}{(\mathrm{RAR})_{(id)[q_{in};\,q_{out}]}\,q_{(in)a}} = \left[\frac{q_{out}}{q_{(out)(f)a}}\right]_{id} = [(\mathrm{O})(\mathrm{OFFA})(\mathrm{RF})]_{(id)[q_{in};q_{out}]}$$

= output – output forcing function amplitude ratio function

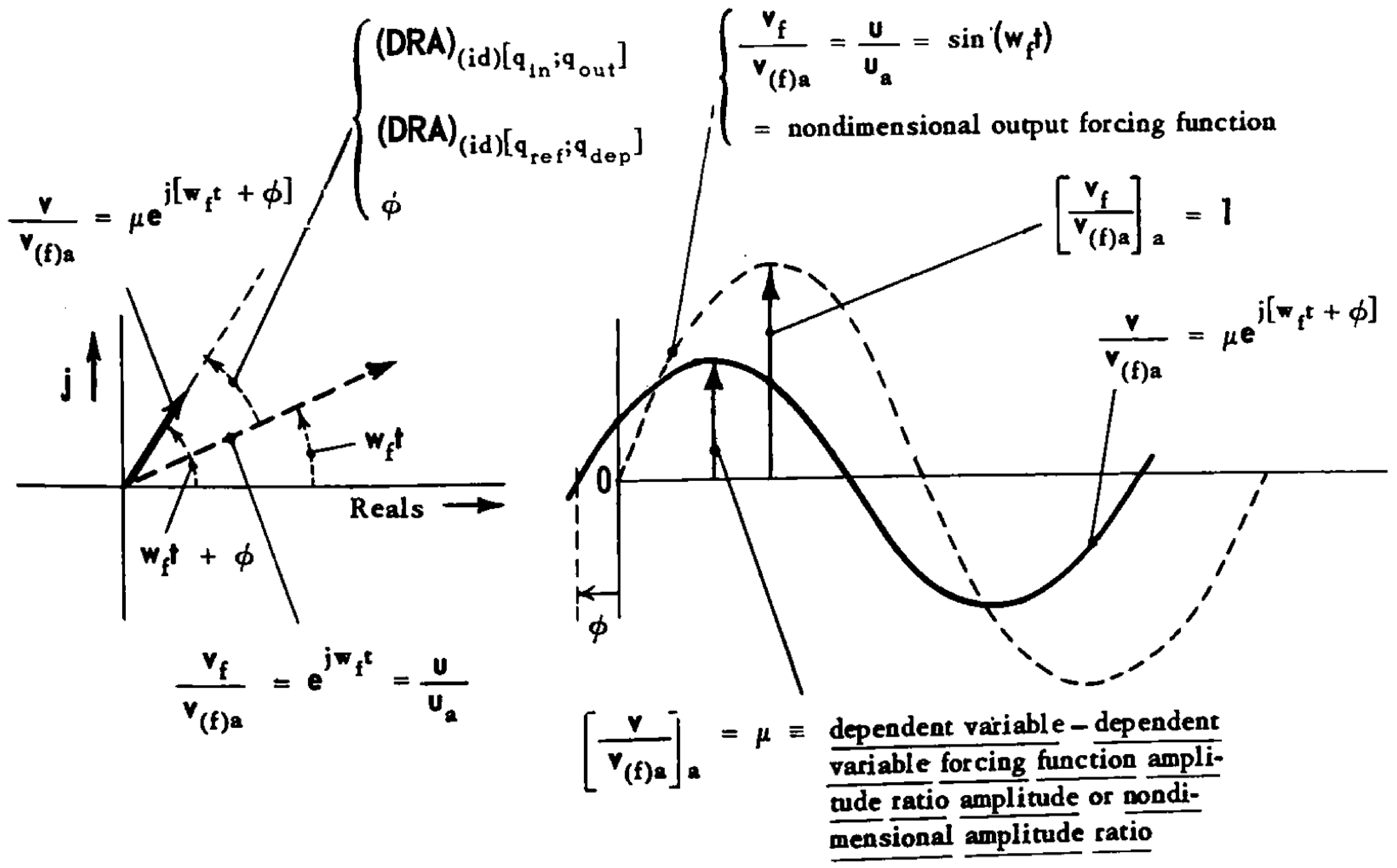
$$= [(\mathrm{DAR})(\mathrm{RAR})\mathrm{R}]_{(id)[q_{in};\,q_{out}]}\,e^{j[w_f t + (\mathrm{DRA})_{(id)[q_{in};q_{out}]}]}$$

Similarly, for the reference variable – dependent variable form

$$\frac{q_{dep}}{(\mathrm{RAR})_{(id)[q_{ref};\,q_{dep}]}\,q_{(ref)a}} = [(\mathrm{DAR})(\mathrm{RAR})\mathrm{R}]_{(id)[q_{ref};\,q_{dep}]}\,e^{j[w_f t + (\mathrm{DRA})_{(id)[q_{ref};q_{dep}]}]} \tag{10}$$

and in working variable form with

$$v_f = \rho_f u \equiv \text{output forcing function}, \qquad \text{then} \quad \frac{v}{v_{(f)a}} = \frac{v}{\rho_f u_a} = \mu e^{j[w_f t + \phi]} \tag{11}$$


$\frac{v}{v_{(f)a}} = \mu e^{j[w_f t + \phi]}$
$(\mathrm{DRA})_{(id)[q_{in};q_{out}]}$
$(\mathrm{DRA})_{(id)[q_{ref};q_{dep}]}$
ϕ
$\frac{v_f}{v_{(f)a}} = \frac{u}{u_a} = \sin(w_f t)$
= nondimensional output forcing function
$\left[\frac{v_f}{v_{(f)a}}\right]_a = 1$
$\frac{v}{v_{(f)a}} = \mu e^{j[w_f t + \phi]}$
j
$w_f t$
Reals
$w_f t + \phi$
0
ϕ
$\frac{v_f}{v_{(f)a}} = e^{j w_f t} = \frac{u}{u_a}$
$\left[\frac{v}{v_{(f)a}}\right]_a = \mu \equiv$ dependent variable – dependent variable forcing function amplitude ratio amplitude or nondimensional amplitude ratio

c) Nondimensional representation of a steady-state sinusoidal response

Definition Summary 13-3. Representation of steady-state sinusoidal response for linear systems.
(Page 3 of 3)

13.15. Section c) of Definition Summary 13-3 gives nondimensional representations of steady-state sinusoidal response for linear systems. The nondimensional response variable is formed as the ratio of the output to the output forcing function. By its definition, the output forcing function always has the dimensions of the output quantity and is the amplitude that the output would have under static or quasi-static conditions, with the reference amplitude ratio taken as a dimension-matching multiplying factor. It is important to note that the nondimensional output forcing function, which is the output forcing function divided by the output forcing function amplitude, is always represented by a rotating vector of unit length or a sine curve of unit amplitude. The corresponding nondimensional response in terms of the output – output forcing function amplitude ratio has an amplitude equal to the dynamic amplitude ratio – reference amplitude ratio ratio, i.e., the nondimensional response amplitude ratio, and is displaced from the nondimensional output forcing function by the dynamic response angle. These relationships are illustrated by the plots of section c) of Definition Summary 13-3.

DEFINITIONS AND NOTATION FOR VARIOUS SINUSOIDAL PERFORMANCE-DESCRIBING FORMS

13.16. In practice, it is useful to have various forms immediately available for representing the steady-state sinusoidal response characteristics of linear operating systems. A number of these forms with the associated definitions and symbols are given in Definition Summary 13-4. For many purposes it is convenient to resolve the generalized performance function form of Eq. (1) of Definition Summary 13-4 into a reference performance function and a corresponding performance function deviation function. The reference function would ordinarily be chosen as some convenient form not too much unlike the actual performance function so that the deviation function would represent a variation of less importance than that corresponding to the reference function. Equation (2) of Definition Summary 13-4 and its accompanying definitions illustrate the combination of a reference performance function and a deviation function to form the performance function. Equations (3), (4), and (5) of Definition Summary 13-4 and the accompanying definitions illustrate the resolution of the reference frequency function into a reference dynamic amplitude ratio and an imaginary exponential with the reference dynamic response angle in its exponent. The performance function deviation function may be broken down into the deviation function amplitude ratio and an imaginary exponential with the dynamic response angle deviation in its exponent.

13.17. Equation (6) of Definition Summary 13-4 and its associated definitions describe an expression for the performance function as the product of an arbitrarily chosen constant reference amplitude ratio and the frequency function, which represents the part of the performance function that depends upon

For practical purposes, it is often convenient to express performance functions in terms of reference performance functions and deviation functions. When the reference variable – dependent variable sinusoidal performance function is taken as the general form,

$$q_{dep} = [PF]_{(id)[q_{ref};q_{dep}]sin}\, q_{ref} \qquad (1)$$

If the performance function is resolved into a reference performance function and a performance function – deviation function,

Note

The subscript "sin" is omitted from the performance function symbols and the word "sinusoidal" is omitted from the names in the remaining expressions of this definition summary. It is understood that all of the forms that occur apply directly to steady-state sinusoidal variations.

$$q_{(dep} = [PF]_{(id)[q_{ref};q_{dep}]ref}\,[(DF)(PF)]_{(id)[q_{ref};q_{dep}]}\, q_{ref} \qquad (2)$$

where

$[PF]_{(id)[q_{ref};q_{dep}]ref}$ = reference variable – dependent variable sinusoidal reference performance function of (id) identified operating system

$[(DF)(PF)]_{(id)[q_{ref};q_{dep}]}$ = reference variable – dependent variable performance function deviation function for (id) identified operating system

The reference performance function and the deviation function may be written as products of dynamic amplitude ratios and imaginary exponentials with dynamic response angle exponents, i.e.

$$[PF]_{(id)[q_{ref};q_{dep}]ref} = (DAR)_{(id)[q_{ref};q_{dep}]ref}\, e^{j[(DRA)_{(id)[q_{ref};q_{dep}]ref}]} \qquad (3)$$

$$[(DF)(PF)]_{(id)[q_{ref};q_{dep}]} = (DFAR)_{(id)[q_{ref};q_{dep}]}\, e^{j[(D)(DRA)]_{(id)[q_{ref};q_{dep}]}} \qquad (4)$$

so that

$$[PF]_{(id)[q_{ref};q_{dep}]} = (DAR)_{(id)[q_{ref};q_{dep}]ref}\,(DFAR)_{(id)[q_{ref};q_{dep}]}\, e^{j(DRA)_{(id)[q_{ref};q_{dep}]ref}}\, e^{j[(D)(DRA)]_{(id)[q_{ref};q_{dep}]}} \qquad (5)$$

$(DAR)_{(id)[q_{ref};q_{dep}]ref}$ = reference quantity – dependent quantity reference dynamic amplitude ratio of (id) identified operating system

$(DRA)_{(id)[q_{ref};q_{dep}]ref}$ = reference quantity – dependent quantity reference dynamic response angle of (id) identified operating system

$(DFAR)_{(id)[q_{ref};q_{dep}]}$ = reference quantity – dependent quantity deviation function amplitude ratio for (id) identified operating system

$[(D)(DRA)]_{(id)[q_{ref};q_{dep}]}$ = reference quantity – dependent quantity dynamic response angle deviation of (id) identified operating system

Definition Summary 13-4. Forms of performance function for describing the sinusoidal response of linear operating systems. (Page 1 of 2)

When the reference performance function is made up of a reference amplitude ratio considered as an arbitrarily chosen constant quantity with dimensions identical with the dimensions of the dynamic amplitude ratio and a frequency function

$$[PF]_{(id)[q_{ref};q_{dep}]} = (RAR)_{(id)[q_{ref};q_{dep}]}(FF)_{(id)[q_{ref};q_{dep}]} \tag{6}$$

$(RAR)_{(id)[q_{ref};q_{dep}]}$ = reference variable – dependent variable reference amplitude ratio of (id) identified operating system (an arbitrarily chosen constant)

$(FF)_{(id)[q_{ref};q_{dep}]}$ = reference variable – dependent variable frequency function of (id) identified operating system

The frequency function may be resolved into a reference frequency function and a frequency function deviation function

$$[PF]_{(id)[q_{ref};q_{dep}]} = (RAR)_{(id)[q_{ref};q_{dep}]}(FF)_{(id)[q_{ref};q_{dep}]ref}[(DF)(FF)]_{(id)[q_{ref};q_{dep}]} \tag{7}$$

$(FF)_{(id)[q_{ref};q_{dep}]ref}$ = reference variable – dependent variable reference frequency function of (id) identified operating system

$[(DF)(FF)]_{(id)[q_{ref};q_{dep}]}$ = reference variable – dependent variable frequency function deviation function of (id) identified operating system

In terms of the dynamic amplitude ratio and the dynamic response angle the performance function is

$$[PF]_{(id)[q_{ref};q_{dep}]} = (DAR)_{(id)[q_{ref};q_{dep}]}e^{j(DRA)_{(id)[q_{ref};q_{dep}]}} \tag{8}$$

$(DAR)_{(id)[q_{ref};q_{dep}]}$ = reference quantity – dependent quantity dynamic amplitude ratio of (id) identified operating system

$(DRA)_{(id)[q_{ref};q_{dep}]}$ = reference quantity – dependent quantity dynamic response angle of (id) identified operating system

If the dynamic amplitude ratio is divided by a reference amplitude ratio, the performance function has the form

$$[PF]_{(id)[q_{ref};q_{dep}]} = (RAR)_{(id)[q_{ref};q_{dep}]}[(DAR)(RAR)R]_{(id)[q_{ref};q_{dep}]}e^{j(DRA)_{(id)[q_{ref};q_{dep}]}} \tag{9}$$

$[(DAR)(RAR)R]_{(id)[q_{ref};q_{dep}]}$ = reference quantity – dependent quantity dynamic amplitude ratio – reference amplitude ratio ratio of (id) identified operating system

The performance function may be expressed in terms of a reference $[(DAR)(RAR)R]$ with a corresponding deviation function

$$\begin{aligned}[PF]_{(id)[q_{ref};q_{dep}]} &= (RAR)_{(id)[q_{ref};q_{dep}]}[(DAR)(RAR)R]_{(id)[q_{ref};q_{dep}]ref}\\ &\quad \times [(DF)[(DAR)(RAR)R]]_{(id)[q_{ref};q_{dep}]}e^{j(DRA)_{(id)[q_{ref};q_{dep}]ref}}e^{j[(D)(DRA)]_{(id)[q_{ref};q_{dep}]}}\end{aligned} \tag{10}$$

where

$[(DAR)(RAR)R]_{(id)[q_{ref};q_{dep}]ref}$ = reference quantity – dependent quantity dynamic amplitude ratio – reference amplitude ratio reference ratio of (id) identified operating system

$[(DF)[(DAR)(RAR)R]]_{(id)[q_{ref};q_{dep}]}$ = reference quantity – dependent quantity dynamic amplitude ratio – reference amplitude ratio ratio deviation function of (id) identified operating system

Definition Summary 13-4. Forms of performance function for describing the sinusoidal response of linear operating systems. (Page 2 of 2)

frequency. Equation (7) and its associated definitions illustrate the resolution of the frequency function into a reference frequency function and the corresponding frequency function deviation function.

13.18. Equation (9) of Definition Summary 13-4 is the usual expression for the performance function in terms of the dynamic amplitude ratio and the dynamic response angle. In general, both of these quantities are functions of forcing frequency. When the dynamic amplitude ratio of Eq. (8) is divided by a reference amplitude ratio, the dynamic amplitude ratio — reference amplitude ratio ratio form of Eq. (9) appears.

13.19. The final form of Definition Summary 13-4 is made up by resolving the dynamic amplitude ratio — reference amplitude ratio ratio into a dynamic amplitude ratio — reference amplitude ratio reference ratio and a dynamic amplitude ratio — reference amplitude ratio ratio deviation function. A corresponding breakdown of the dynamic response angle gives a reference dynamic response angle and a dynamic response angle deviation.

SUMMARY OF SINUSOIDAL PERFORMANCE-DESCRIBING FORMS

13.20. The symbols and definitions of Definition Summary 13-4, supplemented by the symbols and definitions of Definition Summary 13-5 are used to form the equivalent performance function forms that are tabulated in Definition Summary 13-6. The performance functions of Definition Summary 13-6 are divided into eight groups of three forms each. The eight groups correspond to the typical forms of Definition Summary 13-4. The first member of each group particularly describes the input-output performance of an identified operating system with the scalar factors expressed in terms of the input-output sensitivities defined in Definition Summary 13-5. The second member of each group expresses the performance function for the identified operating system in terms of a reference quantity and a dependent quantity that are associated with the actual input and output quantities for the component. The third member of each performance function group gives one or more working variable forms, which are mathematical expressions not specifically associated with the physical quantities involved in the performance of any particular operating system.

13.21. The performance function forms of Definition Summary 13-6 and the definitions of Definition Summaries 13-4 and 13-5 appear in many of the mathematical derivations and engineering studies that occur in Volumes II and III of this book.

POLAR PLANE PLOTS, LINEAR SCALE PLOTS, AND LOGARITHMIC SCALE PLOTS FOR REPRESENTING SINUSOIDAL PERFORMANCE CHARACTERISTICS

13.22. In addition to the complex plane polar plots of Definition Summary 13-2, linear scale and logarithmic scale plots may be used to describe the sinusoidal

$[RF]$ = relating function ; $[RF]_{ref}$ = reference relating function

$[RFRF] = \frac{[RF]}{[RF]_{ref}} = {}'[RF]$ = relating function ratio function, or nondimensional relating function

$(DF)(RF)$ = relating function deviation function ; $(DF)'(RF)$ = nondimensional relating function deviation function

$S_{(id)[q_{in};\, q_{out}]ref}$ = input – output reference sensitivity of (id) identified operating system

$(SDF)_{(id)[q_{in};\, q_{out}]}$ = input – output sensitivity deviation function of (id) identified operating system

$S_{(id)[q_{in};\, q_{out}]}$ = input – output sensitivity of (id) identified operating system

$[S(RS)R]_{(id)(q_{in};q_{out})}$ = sensitivity – reference sensitivity ratio of (id) identified operating system

Note

When the operating system identification subscript and the reference quantity – dependent quantity subscript bracket are omitted, the concept of reference sensitivity S_{ref} is identified with the "K" of conventional servomechanism theory (5). Similarly, when subscripts are omitted, the frequency function is similar to the "G" of conventional servomechanism theory, i.e.,

$$q_{dep}(s) = S_{(id)[q_{ref};\, q_{dep}]ref}\,[FF]_{(id)[q_{ref};\, q_{dep}]}(s)\, q_{ref} \tag{1}$$

$$q_{dep}(s) = \qquad \overset{''}{K} \qquad \overset{''}{G(s)} \qquad q_{ref} \tag{2}$$

In these two equations the variable quantities are shown as functions of the Laplace transform variable* represented by the lower case (s).

---

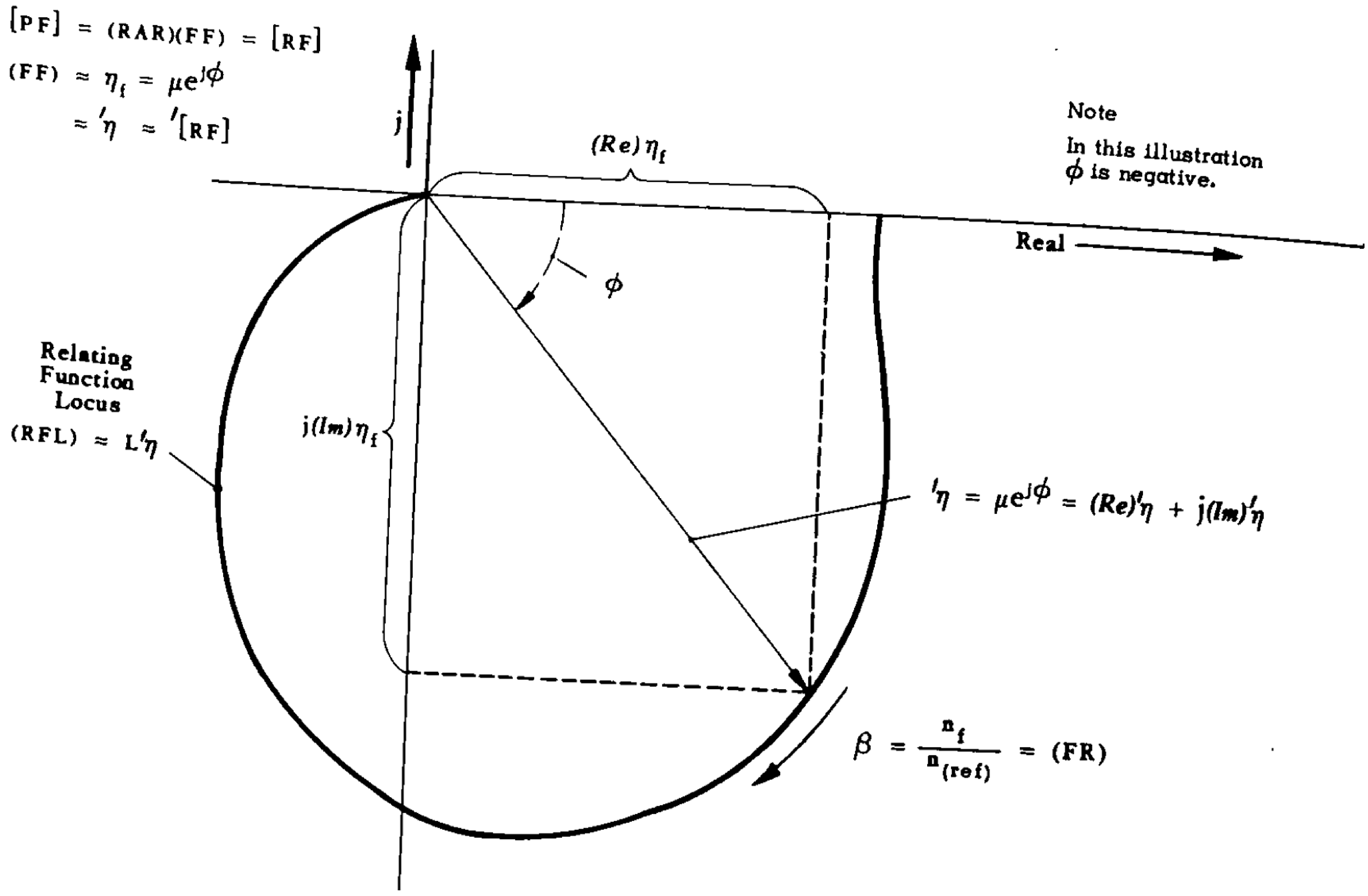
* The theory of the Laplace transform is developed in Vol. I, Transients in Linear Systems, Gardner and Barnes (18).

Definition Summary 13-5. Additional definitions of symbols appearing in Definition Summary 13-6.

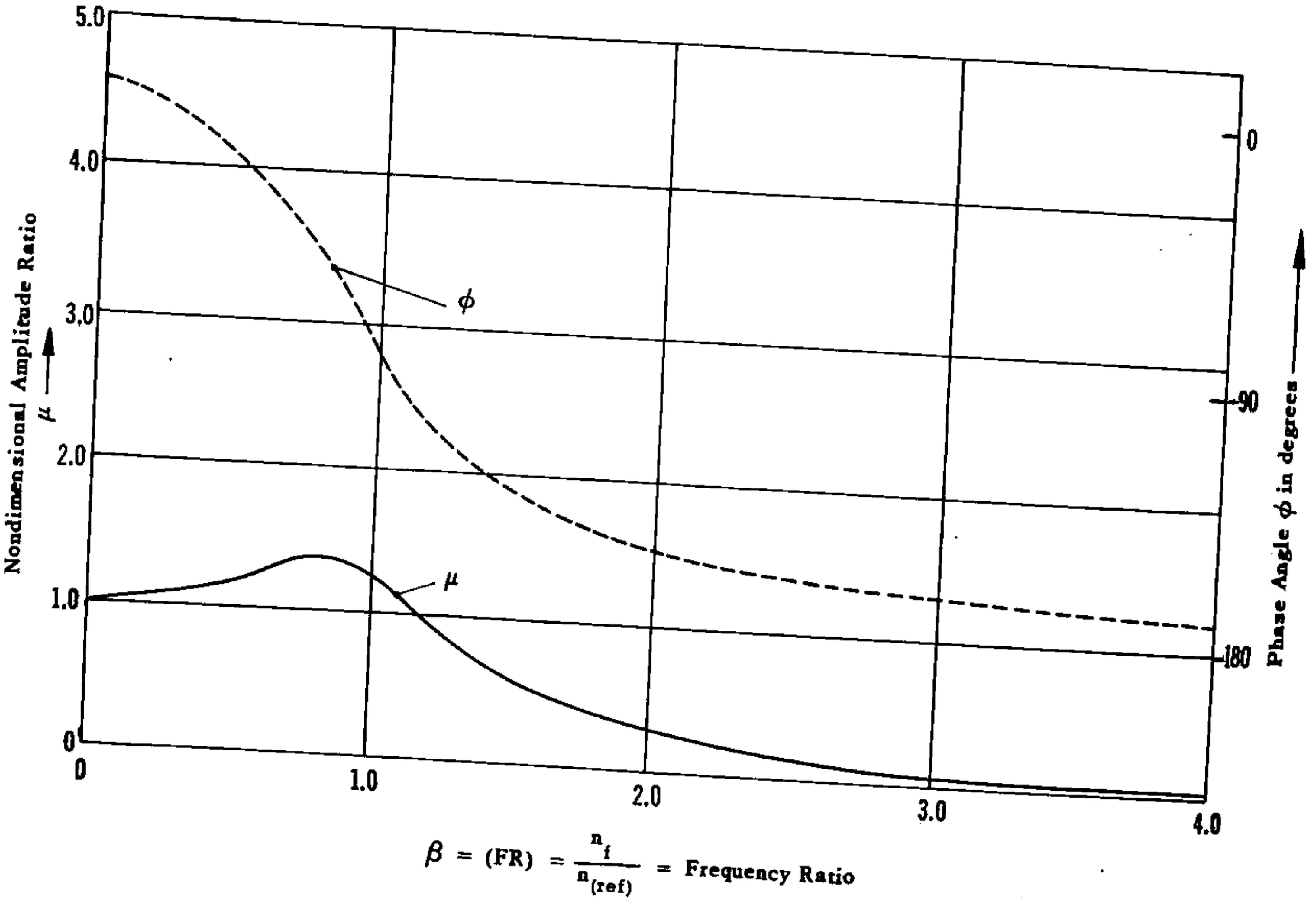
performance characteristics of operating systems. Three methods of representing sinusoidal performance are shown in Fig. 13-1. Maximum effectiveness in data presentation is achieved by using the nondimensional form of Eq. (11) of Definition Summary 13-3 for the performance-describing plots. This form expresses the frequency function of Eqs. (11) and (12) of Definition Summary 13-6 (working symbol: Greek letter eta (η) with subscript f) as the product of the dynamic amplitude ratio – reference amplitude ratio ratio (working symbol: Greek letter mu (μ)) and the imaginary exponential with the dynamic response angle (working symbol: Greek letter phi (ϕ)) in the exponent. For purposes of discussion when no misunderstanding is probable, the quantity represented by the Greek letter mu will be called the nondimensional amplitude ratio, or simply the amplitude ratio, and the dynamic response angle will be called the phase angle,

Variable	Relating Function Forms and Performance-describing Forms	Variable	Eq.
q_{out}	$= [\mathrm{PF}]_{(id)[q_{in};q_{out}]}$	q_{in}	(1)
q_{dep}	$= [\mathrm{PF}]_{(id)[q_{ref};q_{dep}]}$	q_{ref}	(2)
v	$= [\mathrm{RF}]$ or $(\mathrm{RAR})'[\mathrm{RF}]$ or η or $\rho_r{}'\eta$	u	(3)
q_{out}	$= [\mathrm{PF}]_{(id)[q_{in};q_{out}]\mathrm{ref}}[(\mathrm{DF})(\mathrm{PF})]_{(id)[q_{in};q_{out}]}$	q_{in}	(4)
q_{dep}	$= [\mathrm{PF}]_{(id)[q_{ref};q_{dep}]\mathrm{ref}}[(\mathrm{DF})(\mathrm{PF})]_{(id)[q_{ref};q_{dep}]}$	q_{ref}	(5)
v	$= [\mathrm{RF}]_{\mathrm{ref}}[(\mathrm{DF})(\mathrm{RF})]$ or $\eta_r \Delta\eta$	u	(6)
q_{out}	$= S_{(id)[q_{in};q_{out}]\mathrm{ref}}(\mathrm{SDF})_{(id)[q_{in};q_{out}]}e^{j(\mathrm{DRA})_{(id)[q_{in};q_{out}]\mathrm{ref}}}e^{j[(\mathrm{D})(\mathrm{DRA})]_{(id)[q_{in};q_{out}]}}$	q_{in}	(7)
q_{dep}	$= (\mathrm{DAR})_{(id)[q_{ref};q_{dep}]\mathrm{ref}}(\mathrm{DFAR})_{(id)[q_{ref};q_{dep}]}e^{j(\mathrm{DRA})_{(id)[q_{ref};q_{dep}]\mathrm{ref}}}e^{j[(\mathrm{D})(\mathrm{DRA})]_{(id)[q_{ref};q_{dep}]}}$	q_{ref}	(8)
v	$= \rho_r \quad \Delta\rho \quad e^{j\phi_r} \quad e^{j\Delta\phi}$	u	(9)
q_{out}	$= S_{(id)[q_{in};q_{out}]\mathrm{ref}}(\mathrm{FF})_{(id)[q_{in};q_{out}]}$	q_{in}	(10)
q_{dep}	$= (\mathrm{RAR})_{(id)[q_{ref};q_{dep}]\mathrm{ref}}(\mathrm{FF})_{(id)[q_{ref};q_{dep}]}$	q_{ref}	(11)
v	$= \rho_r \quad \eta_f$	u	(12)
q_{out}	$= S_{(id)[q_{in};q_{out}]\mathrm{ref}}(\mathrm{FF})_{(id)[q_{in};q_{out}]\mathrm{ref}}[(\mathrm{DF})(\mathrm{FF})]_{(id)[q_{in};q_{out}]}$	q_{in}	(13)
q_{dep}	$= (\mathrm{RAR})_{(id)[q_{ref};q_{dep}]\mathrm{ref}}(\mathrm{FF})_{(id)[q_{ref};q_{dep}]\mathrm{ref}}[(\mathrm{DF})(\mathrm{FF})]_{(id)[q_{ref};q_{dep}]}$	q_{ref}	(14)
v	$= \rho_r \quad \eta_{fr} \quad \Delta\eta_f$	u	(15)
q_{out}	$= S_{(id)[q_{in};q_{out}]}e^{j(\mathrm{DRA})_{(id)[q_{in};q_{out}]}}$	q_{in}	(16)
q_{dep}	$= (\mathrm{DAR})_{(id)[q_{ref};q_{dep}]}e^{j(\mathrm{DRA})_{(id)[q_{ref};q_{dep}]}}$	q_{ref}	(17)
v	$= \rho \quad e^{j\phi}$	u	(18)
q_{out}	$= S_{(id)[q_{in};q_{out}]\mathrm{ref}}(\mathrm{SRSR})_{(id)[q_{in};q_{out}]}e^{j(\mathrm{DRA})_{(id)[q_{in};q_{out}]}}$	q_{in}	(19)
q_{dep}	$= (\mathrm{RAR})_{(id)[q_{ref};q_{dep}]\mathrm{ref}}[(\mathrm{DAR})(\mathrm{RAR})\mathrm{R}]_{(id)[q_{ref};q_{dep}]}e^{j(\mathrm{DRA})_{(id)[q_{ref};q_{dep}]}}$	q_{ref}	(20)
v	$= \rho_r \quad \mu \quad e^{j\phi}$	u	(21)
q_{out}	$= S_{(id)[q_{in};q_{out}]\mathrm{ref}}(\mathrm{SRSR})_{(id)[q_{in};q_{out}]\mathrm{ref}}e^{j(\mathrm{DRA})_{(id)[q_{in};q_{out}]\mathrm{ref}}}$ $\times [(\mathrm{DF})(\mathrm{SRSR})]_{(id)[q_{in};q_{out}]}e^{j[(\mathrm{D})(\mathrm{DRA})]_{(id)[q_{in};q_{out}]}}$	q_{in}	(22)
q_{dep}	$= (\mathrm{RAR})_{(id)[q_{ref};q_{dep}]\mathrm{ref}}[(\mathrm{DAR})(\mathrm{RAR})\mathrm{R}]_{(id)[q_{ref};q_{dep}]\mathrm{ref}}e^{j(\mathrm{DRA})_{(id)[q_{ref};q_{dep}]\mathrm{ref}}}$ $\times [(\mathrm{DF})[(\mathrm{DAR})(\mathrm{RAR})\mathrm{R}]]_{(id)[q_{ref};q_{dep}]}e^{j[(\mathrm{D})(\mathrm{DRA})]_{(id)[q_{ref};q_{dep}]}}$	q_{ref}	(23)
v	$= \underbrace{\rho_r \quad \mu_r \quad e^{j\phi_r}} \quad \underbrace{(\Delta\mu) \quad e^{j\Delta\phi}}$	u	(24)
v	$= \quad \eta_r \quad (\Delta\eta)$	u	(25)
v	$= \rho_r \quad {}'\eta_r \quad \Delta'\eta$	u	(26)

Definition Summary 13-6. Performance function forms for steady-state conditions with sinusoidal independent quantity variations.

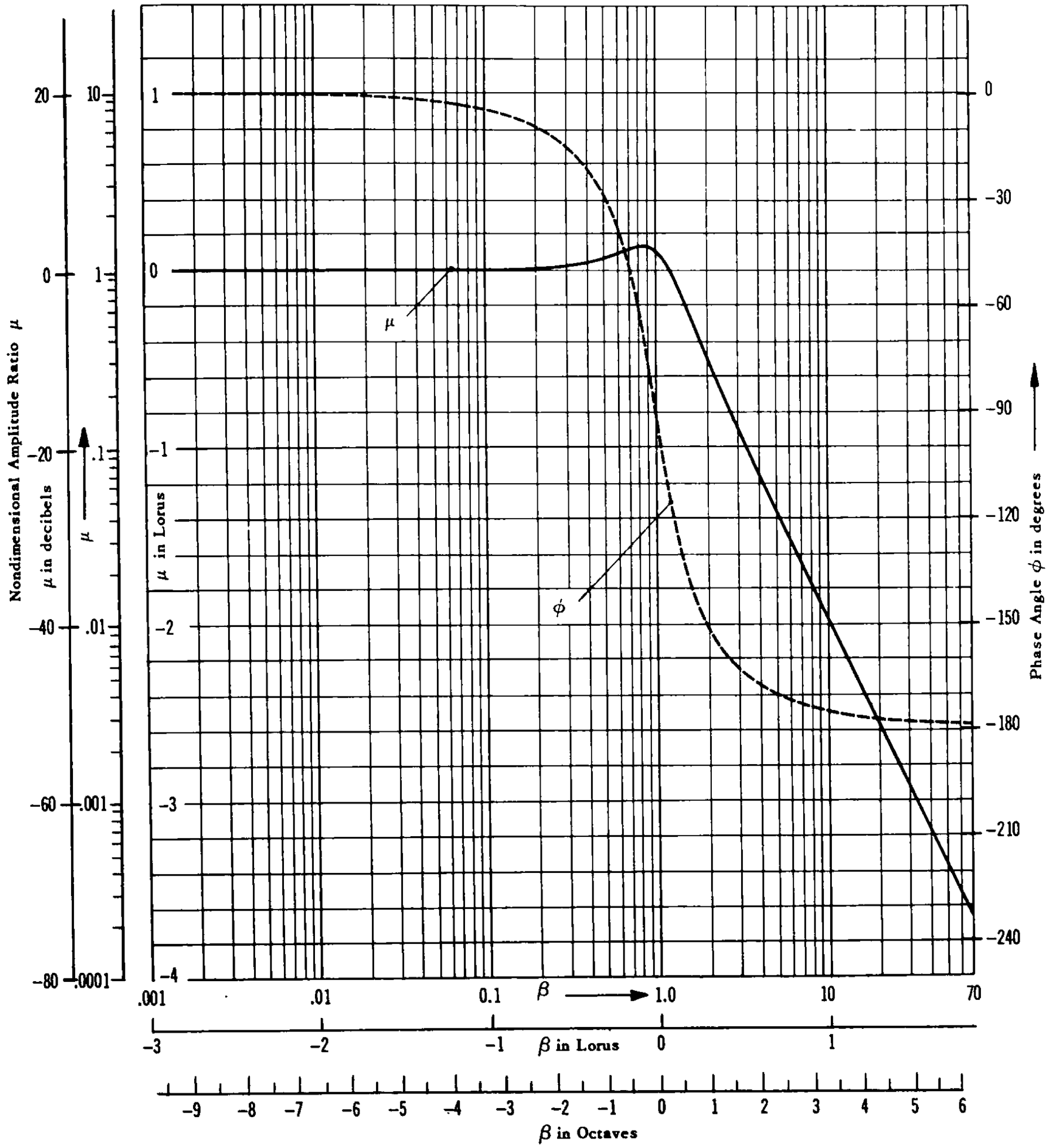


a) Complex plane representation of sinusoidal response characteristics



b) Linear scale sinusoidal response characteristics

Fig. 13-1. Representation of sinusoidal response characteristics. (Page 1 of 2)



c) Logarithmic scale sinusoidal response characteristics

Fig. 13-1. Representation of sinusoidal response characteristics. (Page 2 of 2)

or the phase. The independent variable that determines the amplitude ratio and the phase angle as dependent variables is generally the frequency ratio (symbol: Greek letter beta (β)) or the characteristic time – forcing period ratio* (symbol: Greek letter alpha (α)).

13.23. Figure 13-1a is a complex plane polar diagram showing the real and imaginary parts of the frequency function as projections on the real and imaginary complex plane axes of a vector with a length proportional to the nondimensional amplitude ratio and making an angle equal to the phase angle with the positive direction of the real axis.

13.24. The performance locus for the frequency function is the continuous curve traced out by the end points of the frequency function vector as the frequency ratio varies from zero to infinity. The direction along the locus corresponding to increasing values of frequency ratio is indicated by an arrow parallel to the curve.

13.25. Figure 13-1b illustrates the representation of the nondimensional frequency function by means of a nondimensional amplitude ratio curve and a phase angle curved placed on linear coordinates with frequency ratio as the independent variable. The frequency function representation of Fig. 13-1c is similar to that of Fig. 13-1b except that logarithmic scales are used for the amplitude ratio and frequency ratio plots. Logarithmic ratio units (loru) for base 10 logarithms and octaves (logarithms for base 2) are both shown for the frequency ratio scale. Base 10 logarithmic ratio units and decibels (twenty times base 10 logarithms) are shown for the amplitude ratio scale. Comparison of Figs. 13-1b and 13-1c shows that logarithmic scales simplify the shape of the amplitude ratio curve in the sense that for high frequency ratio values the plot becomes a downward-sloping straight line instead of a curve asymptotic to the abscissa axis. Logarithmic coordinates also make it possible to include ranges of the variables that are much greater than the ranges possible on a plot of given size when linear scales are used.

COMPLEX PLANE REPRESENTATION OF PRODUCT COMBINATIONS OF FREQUENCY FUNCTIONS

13.26. Figure 13-2 illustrates the product combination of two nondimensional relating functions to give a third nondimensional relating function. In general, the nondimensional amplitude ratio of the resultant is the product of the amplitude ratios of the components, and the phase angle of the resultant is the algebraic sum of the phase angles of the components. This situation is shown in Fig. 13-2a for the complex plane representation of two illustrative nondimensional relating functions and their product resultant. The first relating function is shown as a function of a frequency ratio with an arbitrarily selected reference frequency. The second relating function is shown as a function of a characteristic time — forcing period ratio. For the illustrative plots, it is assumed that the reference period — characteristic time ratio* is equal to one-half. The locus representing the resultant in Fig. 13-2a is shown by a heavy line with the frequency ratio for the first component as the independent variable. Typical vectors and symbols for each component and the resultant are given in the figure. It is noted in the figure that division of one nondimensional relating function by another is equivalent to the product combination described by the diagram using the reciprocal of the vector divisor and the negative of the corresponding angle in its associated imaginary exponential.

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* The significance of this ratio will not appear until the first-order differential equation derivation of Vol. II is completed. The exact meaning of alpha is not important for the purposes of the present discussion.

LINEAR SCALE REPRESENTATION OF FREQUENCY FUNCTION PRODUCTS

13.27. Figure 13-2b represents two nondimensional relating functions, which in this case are identical with the frequency functions of the corresponding performance functions, and their product-type resultant in terms of amplitude ratio and phase angle curves plotted in terms of linear coordinate scales. The phase angle of the resultant is the algebraic sum of the phase angles of the components, and the amplitude ratio of the resultant is the product of the amplitude ratios of the components. In forming the resultant, the rule of phase angle combination is simple addition that may be carried out graphically on the plot. On the other hand, the resultant amplitude ratio must be found by a multiplication of values read from the amplitude ratio curves for the components.

LOGARITHMIC SCALE REPRESENTATION OF FREQUENCY FUNCTION PRODUCTS

13.28. Figure 13-2c is similar to Fig. 13-2b except that linear scales are replaced by logarithmic scales for frequency ratio and amplitude ratio. Because the multiplication of two numbers corresponds to the addition of their logarithms, the resultant amplitude ratio curve on the plot of Fig. 13-2c may be found by graphical addition of the amplitude ratio curves for the component frequency functions. This addition may be carried out on the diagram itself without tabulations of numbers. The antilog number scales superimposed on the logarithmic scales make it possible to read numerical values of any variable as conveniently as if linear scales were used. The ease with which graphical methods may be applied to find resultant frequency functions from component frequency functions gives the logarithmic scale plot a considerable advantage over other types of diagrams for representing functions of this type.

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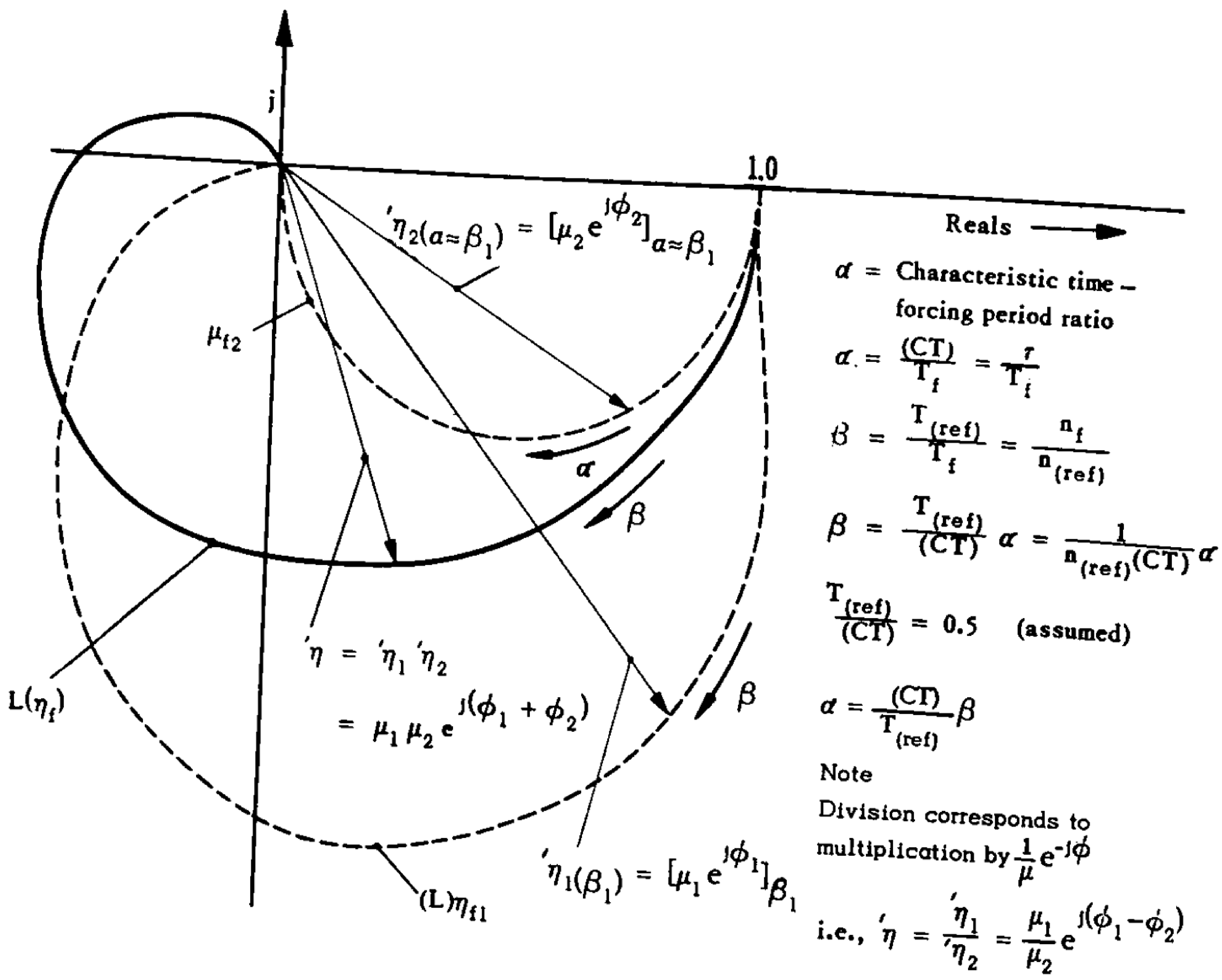
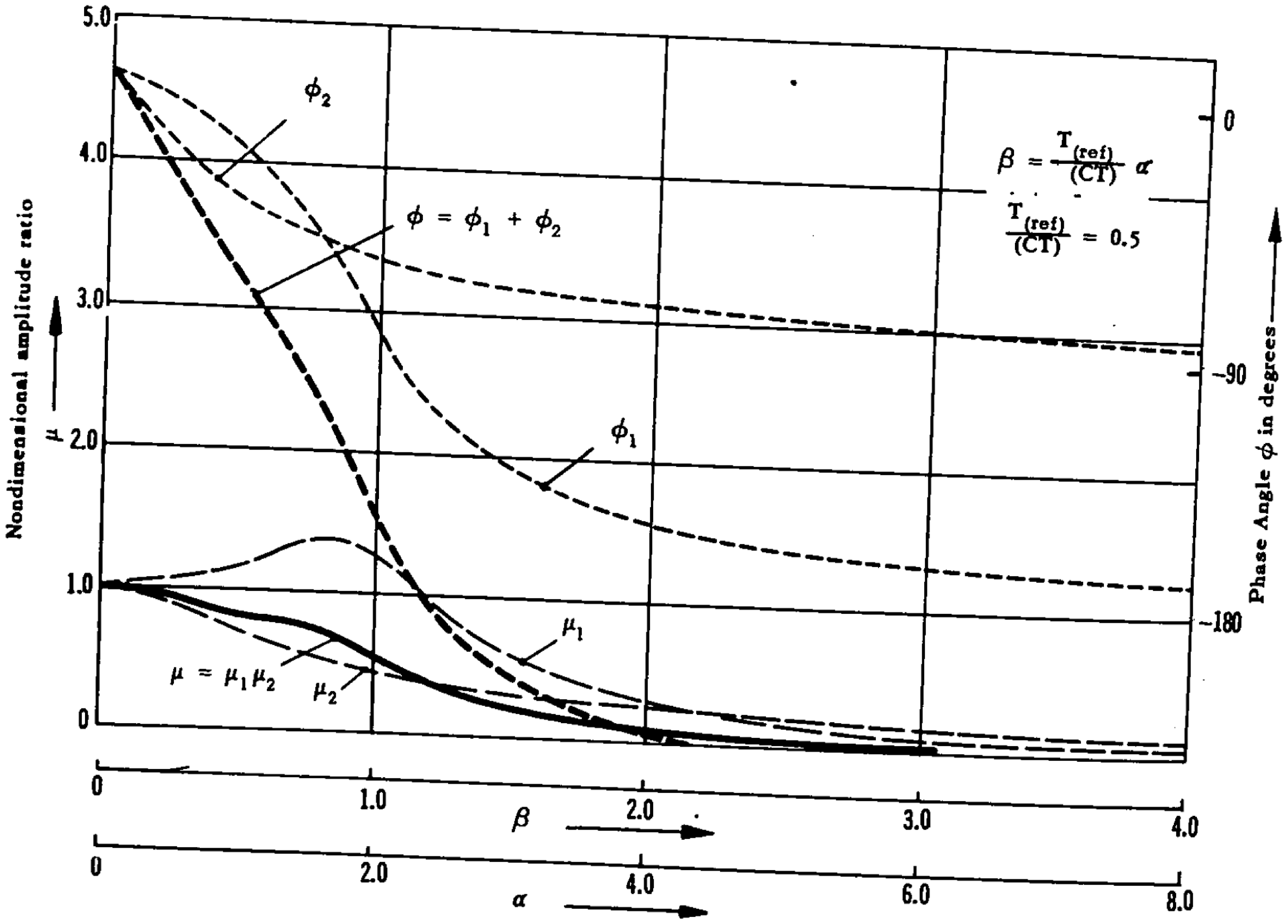
* Ratios of this type must always be determined whenever frequency functions based on different reference frequencies or characteristic times are to be combined.

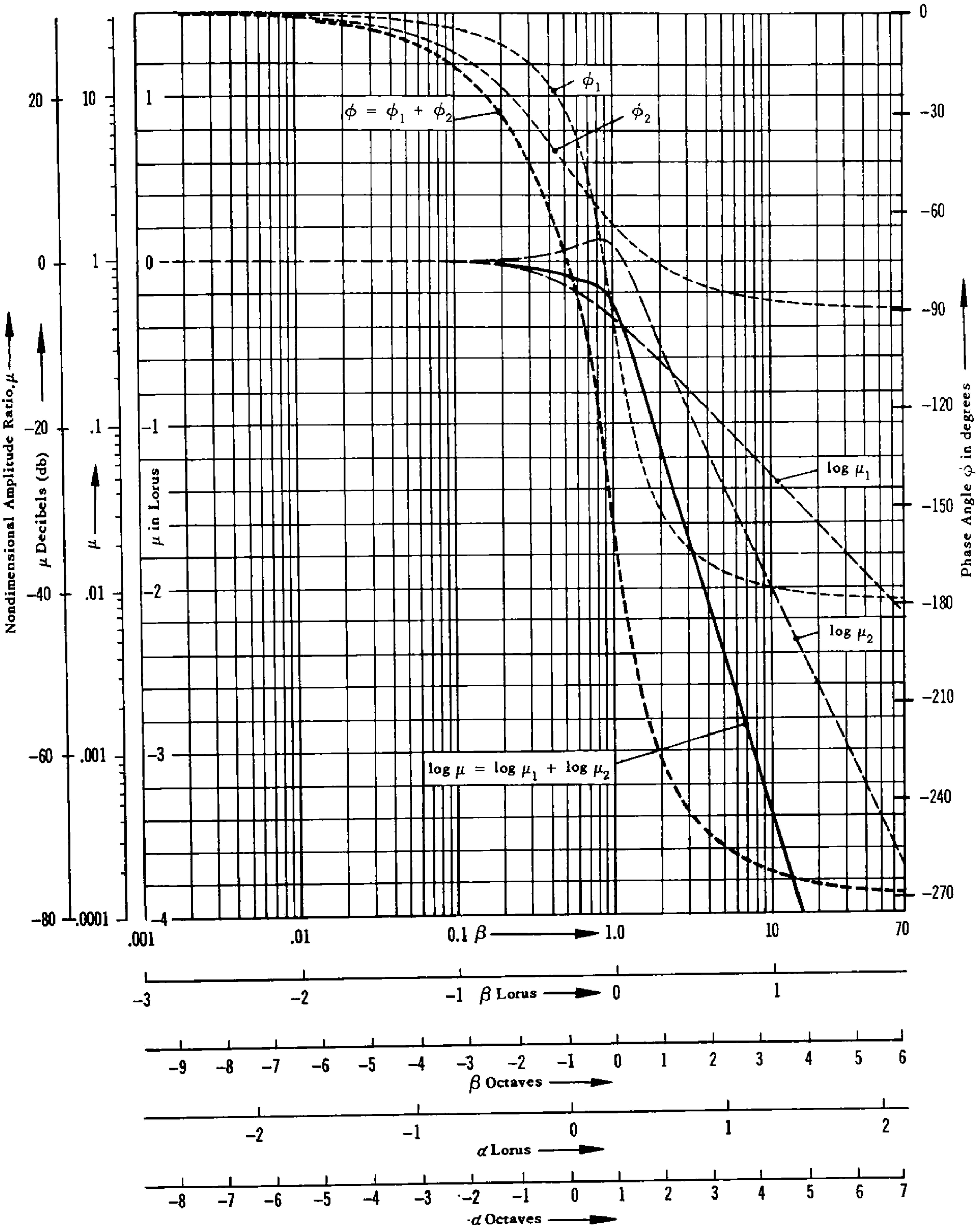

j
1.0
Reals
$'\eta_2(\alpha \approx \beta_1) = [\mu_2 e^{j\phi_2}]_{\alpha \approx \beta_1}$
μ_{f2}
α
β
β
$L(\eta_f)$
$'\eta = {'\eta_1}\,{'\eta_2}$
$= \mu_1 \mu_2 e^{j(\phi_1 + \phi_2)}$
$'\eta_1(\beta_1) = [\mu_1 e^{j\phi_1}]_{\beta_1}$
$(L)\eta_{f1}$
α = Characteristic time – forcing period ratio
$\alpha = \frac{(CT)}{T_f} = \frac{\tau}{T_f}$
$\beta = \frac{T_{(ref)}}{T_f} = \frac{n_f}{n_{(ref)}}$
$\beta = \frac{T_{(ref)}}{(CT)}\alpha = \frac{1}{n_{(ref)}(CT)}\alpha$
$\frac{T_{(ref)}}{(CT)} = 0.5$ (assumed)
$\alpha = \frac{(CT)}{T_{(ref)}}\beta$
Note
Division corresponds to multiplication by $\frac{1}{\mu}e^{-j\phi}$
i.e., $'\eta = \frac{'\eta_1}{'\eta_2} = \frac{\mu_1}{\mu_2}e^{j(\phi_1 - \phi_2)}$

a) Complex plane representation for product combination of sinusoidal response characteristics



b) Linear scale representation for product combination of sinusoidal response characteristics

Fig. 13-2. Representations for combination of sinusoidal response characteristics. (Page 1 of 2)



c) Logarithmic scale representation for product combination of sinusoidal response characteristics

Fig. 13-2. Representations for combination of sinusoidal response characteristics. (Page 2 of 2)

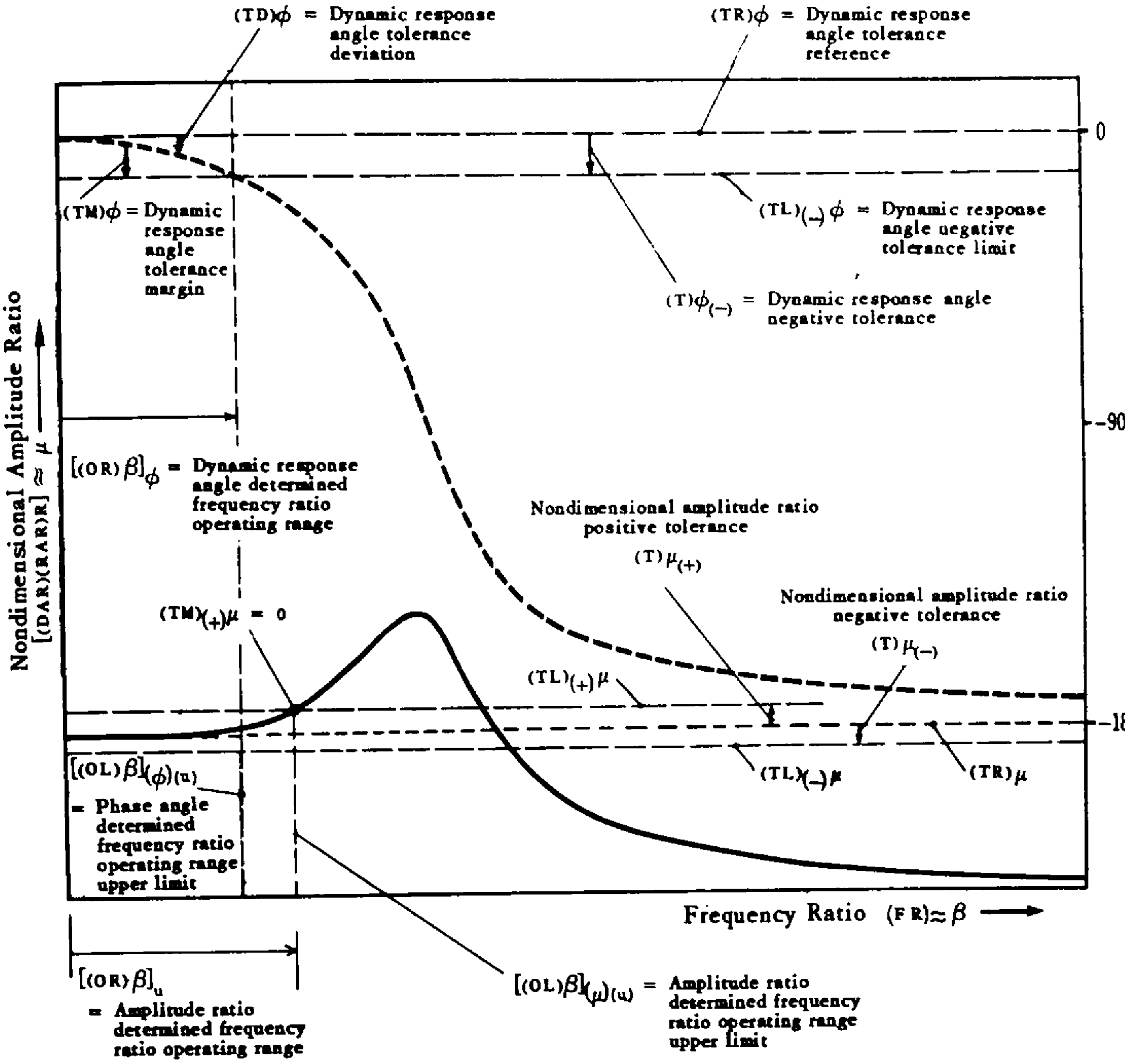
13.29. In practice, logarithmic scale plots of nondimensional amplitude ratios may be transformed into amplitude ratio plots for actual performance functions by adding the logarithm of the reference amplitude ratio to the resultant frequency function plot. Because in logarithmic terms this change is carried out by the addition of a constant, it may be accomplished by the simple process of relabeling the logarithmic amplitude ratio scale. Many examples of this procedure appear later in this book.

OPERATING FREQUENCY RANGE AND BAND WIDTH

13.30. Specifications for operating system performance are often written in terms of frequency function amplitude ratio and phase angle. For example, the operating frequency range for a given component might be described as the frequency band over which the nondimensional frequency ratio does not deviate more than plus or minus one-tenth from unity. The operating frequency range might also be designated as the frequency band within which the phase angle remains within a tolerance limit (see par. 9.28) of minus twenty degrees.

13.31. Figure 13-3 illustrates an application of the tolerance-describing concepts defined in paragraph 9.28 to the sinusoidal performance of an operating system. The frequency function of the system used for illustration purposes is represented by the nondimensional amplitude ratio and the phase angle plotted as functions of the frequency ratio. In this diagram, the amplitude ratio tolerance reference is taken as a constant level equal to the zero-frequency level of this ratio. The phase angle tolerance reference is taken as zero. Upper and lower tolerance limits are shown as equally spaced about the tolerance reference. The amplitude ratio determined frequency ratio operating range upper limit (symbol $[(OL)\beta]_{(\mu)u}$) is the frequency ratio for which the amplitude ratio upper tolerance margin becomes zero. No corresponding lower tolerance margin exists for a low-frequency limit because of the shape of the amplitude ratio curve. The amplitude ratio determined frequency ratio operating range (symbol $[(OR)\beta]_{(u)}$) includes the band of frequency ratios from zero up to the high-frequency ratio operating limit. This frequency ratio range is also called the operating frequency ratio band width, or, when no misunderstanding is possible, simply the band width. It is obvious that the band width for any given case depends upon the specified tolerance limits. The band width becomes greater as the tolerance is made wider.

13.32. In Fig. 13-3, a frequency ratio operating limit corresponding to the zero phase angle tolerance margin is shown as differing from the amplitude ratio determined operating limit. Differences of this kind between operating limits determined by various aspects of system operation are common in practice. Many examples that use the concepts of tolerance description are discussed in Volume III of this book.



Note

The *frequency ratio operating range* is also defined as the *operating frequency band, (symbol (OFB))* or the *operating frequency band width (symbol (OFBW))* or when no misunderstanding is possible the *band width (symbol (BW)).*

Fig. 13-3. Operating frequency ratio range and band width.

COMBINATION OF SINUSOIDAL PERFORMANCE FUNCTIONS TO DESCRIBE OPERATION OF SYSTEMS WITH MULTIPLE INPUTS

13.33. In practice, operating systems may receive an indefinite number of inputs and produce some other indefinite number of outputs. Of the various methods available for writing performance functions to specify the action of a system with multiple inputs and outputs, one of the most effective schemes is to use a summation of sinusoidal performance-describing forms. The general features of this procedure are outlined in Definition Summary 13-7. One of the

In general

$$q_{(out)(j)} = [\mathbf{PF}]_{(id)[q_{(in)(1)},\ldots;q_{(out)(j)}]}[q_{(in)(1)};q_{(in)(2)};\ldots] \tag{1}$$

In terms of performance functions and input function products (refer to Definition Summary 13-1)

$$q_{(out)(j)} = \sum_{(i)} [\mathbf{PF}]_{(id)[(IFP)_{(i)};q_{(out)(j)}]}(\mathbf{IFP})_{(i)} \tag{2}$$

Note

Summation over i must include all terms necessary to completely express $q_{(out)(j)}$. The input function products are made up of products of functions of the inputs.

In terms of reference sensitivities and frequency functions

$$q_{(out)(j)} = \sum_{(i)} \mathbf{S}_{(id)[(IFP)_{(i)};q_{(out)(j)}](ref)}(\mathbf{FF})_{(id)[(IFP)_{(i)};q_{(out)(j)}]}\underline{(\mathbf{IFP})_{(i)}} \tag{3}$$

In terms of sensitivity ratios and frequency function ratios

$$\begin{aligned} q_{(out)(j)} = {} & \mathbf{S}_{(id)[(IFP)_{(ref)};q_{(out)(j)}](ref)}(\mathbf{FF})_{(id)[(IFP)_{(ref)};q_{(out)(j)}]} \\ & \times \{(\mathbf{IFP})_{(ref)} + \sum_{(k)} (\mathbf{SR})_{(id)[(IFP)_{(k)};(IFP)_{(ref)}]}(\mathbf{FFR})_{(id)[(IFP)_{(k)};(IFP)_{(ref)}]}(\mathbf{IFP})_{(k)}\} \end{aligned} \tag{4}$$

Note

Summation over k must include all terms necessary to completely express $q_{(out)(j)}$. The reference input function product $(\mathbf{IFP})_{(ref)}$ is chosen for convenience and effectiveness in use of the performance equation.

where by definition

$$(\mathbf{SR})_{(id)[(IFP)_{(k)};(IFP)_{(ref)}]} = \frac{\mathbf{S}_{(id)[(IFP)_{(k)};q_{(out)(j)}]}}{\mathbf{S}_{(id)[(IFP)_{(ref)};q_{(out)(j)}](ref)}} \equiv \text{k^{th} input function product – reference input function product sensitivity ratio}$$

$$(\mathbf{FFRF})_{(id)[(IFP)_{(k)};(IFP)_{(ref)}]} = \frac{(\mathbf{FF})_{(id)[(IFP)_{(k)};q_{(out)(j)}]}}{(\mathbf{FF})_{(id)[(IFP)_{(k)};q_{(out)(j)}](ref)}} \equiv \text{k^{th} input function product – reference input function frequency function ratio function}$$

$$\mathbf{S}_{(id)[(IFP)_{(ref)};q_{(out)(j)}](ref)} \equiv \text{Reference input function product – j^{th} output reference sensitivity for (id) identified operating component}$$

$$(\mathbf{FF})_{(id)[(IFP)_{(ref)};q_{(out)(j)}](ref)} \equiv \text{Reference input function product – j^{th} output reference frequency function for (id) identified operating component}$$

a) Performance equation forms for j^{th} output of (id) identified operating component

$$\begin{aligned} [\mathbf{PF}]_{(id)[q_{(in)(1)},\ldots;q_{(out)(j)}]} &= \sum_{(i)} [\mathbf{PF}]_{(id)[(IFP)_{(i)};q_{(out)(j)}]}[\ ]_{(i)} \\ &= \sum_{(i)} \mathbf{S}_{(id)[(IFP)_{(i)};q_{(out)(j)}]}(\mathbf{FF})_{(id)[(IFP)_{(i)};q_{(out)(j)}]}[\ ]_{(i)} \\ &= \mathbf{S}_{(id)[(IFP)_{(ref)};q_{(out)(j)}]}(\mathbf{FF})_{(id)[(IFP)_{(ref)};q_{(out)(j)}]} \\ &\quad \times \{[\ ]_{(ref)} + \sum_{(k)} (\mathbf{SR})_{(id)[(IFP)_{(k)};(IFP)_{(ref)}]}(\mathbf{FFRF})_{(id)[(IFP)_{(k)};(IFP)_{(ref)}]}[\ ]_{(k)}\} \end{aligned} \tag{5}$$

Note

An open pair of brackets with a subscript indicates the location in the performance equation of the argument variable identified in the subscripts of the main component symbols.

b) Scalar performance operator forms in terms of sensitivity ratio and frequency function ratio functions

Definition Summary 13-7. Defining relationships for sensitivity ratio and frequency function ratio function.

several system outputs is designated as the j^{th} output and used for illustrative purposes. It is understood that any number of outputs may be taken into account by repeating the steps applied for describing the relationships between the j^{th} output and its associated inputs.

13.34. Equation (2) of Definition Summary 13-7 shows the j^{th} output expressed as a summation of performance functions, each of which depends upon an input function product of the type described in section b) of Definition Summary 13-1. The typical performance function and its associated input function product are designated by (i) subscripts. For the form of Eq. (2) of Definition Summary 13-7 to properly represent the j^{th} output, the summation must be taken over all the necessary integer values of (i). Equation (3) of Definition Summary 13-7 is a form equivalent to Eq. (2) with the typical performance function of the summation written as the product of a reference sensitivity and a frequency function to indicate that the j^{th} output is expressed in terms of the sinusoidal performance characteristics of the operating system considered.

13.35. For practical engineering applications, it is useful to select one input function product as the reference input function product and to take the corresponding reference sensitivity and frequency function outside the summation of sinusoidal performance-describing terms. The result of this procedure is shown in Eq. (4) of Definition Summary 13-7 and its associated definitions. The reference sensitivity for the reference input function product and the associated frequency function, which becomes the reference frequency function, appears outside a bracket. The first term of this bracket is the reference input function product itself, and the second term is a summation of terms each formed as the product of a sensitivity ratio, a frequency function ratio function, and the corresponding input function product. By definition, the k^{th} input function product – reference input function product sensitivity ratio is the ratio of the k^{th} input function product reference sensitivity to the reference input function product reference sensitivity. The k^{th} input function product – reference input function product reference frequency function ratio function is the ratio of the k^{th} input function product frequency function to the reference input function product reference frequency function. Section a) of Definition Summary 13-7 shows the essential relationships among the concepts described above and gives the corresponding symbols. Section b) of Definition Summary 13-7 brings together various forms of the performance function associated with the j^{th} output from an identified operating system.

13.36. It is important to notice that, in forming sensitivity ratios and frequency function ratio functions, the reference input function product appears in the compared quantity subscript position and the identified input function product appears in the reference quantity subscript position. This means that the reference quantity – compared quantity positions are occupied by the input quantities that remain in the ratios after the output quantity has been cancelled out, rather than by the reference sensitivities that appear as denominators and numerators of sensitivity ratios. By use of this subscript convention for sensitivity ratios, their essential physical meaning is stressed as the magnitude of the reference input function product equivalent to a unit of the identified input function product so far as output effects are concerned. For the same reasons, the reference input function product is placed in the compared quantity subscript position, and the identified input function product is given the reference quantity subscript position in the subscripts for the frequency function ratio function.

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